Kick stability in groups and dynamical systems

Leonid Polterovich * and Zeev Rudnick
School of Mathematical Sciences
Tel Aviv University
69978 Tel Aviv, Israel
polterov@math.tau.ac.il rudnick@math.tau.ac.il

February 1, 2008

Abstract

We consider a general construction of "kicked systems" which extend the framework of classical dynamics. Let G be a group of measure preserving transformations of a probability space. Given a one-parameter/cyclic subgroup (the flow), and any sequence of elements (the kicks) we define the kicked dynamics on the space by alternately flowing with given period, then applying a kick. Our main finding is the following stability phenomenon: the kicked system often inherits recurrence properties of the original flow. We present three main examples.

1) G is the torus. We show that for generic linear flows, and any sequence of kicks, the trajectories of the kicked system are uniformly distributed for almost all periods.

2) G is a discrete subgroup of $PSL(2, \mathbb{R})$ acting on the unit tangent bundle of a Riemann surface. The flow is generated by a single element of G, and we take any bounded sequence of elements of G as our kicks. We prove that the kicked system is mixing for all sufficiently large periods if and only if the generator is of infinite order and is not conjugate to its inverse in G.

3) G is the group of Hamiltonian diffeomorphisms of a closed symplectic manifold. We assume that the flow is rapidly growing in the sense of Hofer's norm, and the kicks are bounded. We prove that for a positive proportion of the periods the kicked system inherits a kind of energy conservation law and is thus super-recurrent.

We use tools of geometric group theory (quasi-morphisms) and symplectic topology (Hofer's geometry).

Key words: sequence of transformations, kicked system, stability, time-reversing symmetry, uniform distribution, lattice, quasi-morphism, Hamiltonian diffeomorphism, Hofer's metric.

MSC2000: Primary 37Axx, Secondary 11K06, 20F69, 53Dxx, 70Kxx

*Supported by THE ISRAEL SCIENCE FOUNDATION founded by the Israel Academy of Sciences and Humanities.

1 Introduction and main results

In the present paper we address the following question: "How far can a flow be kicked?". More precisely, consider the behavior of a one parameter/cyclic subgroup of a Lie group under the influence of a sequence of kicks. The kicks arrive periodically in time. The kicks are deterministic, while the period is chosen "at random". We are interested in the following stability type question: does the kicked system inherit some recurrence properties of the original one? It turns out that in some situations (linear flows on tori, isometries of $PSL(2,\mathbb{R})/\Gamma$, "rapidly growing" Hamiltonian flows on symplectic manifolds) such a stability indeed takes place with positive probability even when the kicks are quite large.

1.1 Sequential systems

Let G be a group. Consider the set $G^\infty = G\times G\times ...$ of all infinite sequences $f_* = \{f_i\}, i \in \mathbb{N}$. Given an action of G on a set X, one can view f_* as a dynamical system (see [BB]): The trajectory of a point $x \in X$ is defined as $\{x_i\}$, where $x_0 = x,\ x_1 = f_1 x,\ x_2 = f_2 f_1 x$ and so on. We write $f^{(i)}$ for the *evolution* of f_* given by $f_i f_{i-1} ... f_1$ and set $f^{(0)} = \mathbb{1}$, so $x_i = f^{(i)}x$. Note that the constant sequence $f_* = \{f\}$ generates iterations of a single map f.

Sequences of maps provide a natural framework for the study of perturbations of the usual settings of dynamical systems [1]. We start with the following simple model. Write $\mathcal{T}$ for either $\mathbb{R}$ or $\mathbb{Z}$. Let $(h^t),\ t \in \mathcal{T}$ be a one-parameter or cyclic subgroup of G which represents a dynamical system on X. Assume that the system is influenced by a sequence of kicks $\{\phi_i\} \in G^\infty$. The kicks arrive periodically in time with some positive period $\tau \in \mathcal{T}$. The kicked dynamics is described by a sequential system ${f_*}^\tau = \{\phi_i h^\tau\}$. An orbit $\{x_i\}$ of the kicked system looks as follows: In order to get x_i from x_{i-1}, go with the flow for the time τ and then apply the i-th kick: $x_i = \phi_i h^\tau x_{i-1}$.

One cannot expect general sequential systems to possess some interesting dynamical properties. For instance if G acts by measure preserving transformations, a sequential system may violate the Poincare recurrence theorem etc. However, the kicked systems described above are very special.

Informal Definition 1.1.A. A dynamical property of a subgroup (h^t) is called *kick stable*, if for every sequence of kicks $\{\phi_i\}$ (possibly satisfying some mild assumptions) the kicked system ${f_*}^\tau$ inherits this property for a large set of periods τ.

In the present paper we discuss various examples and counter-examples to kick stability. Before presenting a formal definition (see 1.3.A below) let us specify some dynamical properties we wish to deal with and consider a number of examples. The definitions below (possibly with exception of 1.1.C) are straightforward extensions of standard dynamical notions to the framework of sequential systems. Assume that G is a topological group which acts by measure preserving homeomorphisms on a topological space X with a Borel probability measure μ. In 1.1.B - 1.1.D below X is assumed to be compact. Let $f_* \in G^\infty$ be a sequential system.

1.1.B. (see [P3]) A sequential system f_* is called *strictly ergodic* if for every continuous

[1] In particular, they are used in various models of random dynamics [Ki], see 1.8 below for further comments.

function F on X the Birkhoff sums

$$\frac{1}{N}\sum_{i=0}^{N-1} F\circ f^{(i)}$$

converge uniformly to $\int_X F(x)d\mu(x)$.

1.1.C. A continuous function F is called *a quasi-integral* of f_* if

$$\limsup_{N\to\infty} \max_X \frac{1}{N}\sum_{i=0}^{N-1} F\circ f^{(i)} > \int_X F(x)d\mu(x).$$

Clearly these definitions complement each other, that is either f_* is strictly ergodic, or it admits a quasi-integral. Sometime it is useful to relax the continuity assumption and to work with characteristic functions χ_A of measurable subsets $A\subset X$.

1.1.D. If $F=\chi_A$ satisfies 1.1.B we say that A is *ideally recurrent* for f_*, and if $F=\chi_A$ satisfies 1.1.C then A is called *super-recurrent* for f_*. Informally speaking, ideal recurrence means that trajectories of the system visit A with the frequency $\mu(A)$, while super-recurrence means that there exist arbitrarily long finite pieces of trajectories of f_* which visit A with the frequency $>\mu(A)$.

1.1.E. A system f_* is called *mixing* if for any two L^2-functions F and G on X the sequence

$$\int_X F(f^{(i)}x)G(x)d\mu(x)$$

converges to

$$\int_X F(x)d\mu(x)\int_X G(x)d\mu(x).$$

1.2 Examples of kick stable flows

Example 1.2.A (stable uniform distribution). Suppose that G is the circle $S^1=\mathbb{R}/\mathbb{Z}$ which acts on itself by shifts. Consider the flow

$$h^t: S^1\to S^1,\ h^t x = x+t \mod 1.$$

A characteristic feature of this flow is that for irrational values of τ, every trajectory of the sequential system $\{h^\tau\}$ is uniformly distributed in S^1, which in our language means that every open interval of S^1 is ideally recurrent. Take an arbitrary sequence of kicks $\phi_*\in(S^1)^\infty$, and consider the kicked system ${f_*}^\tau=\{\phi_i h^\tau\}$.

Theorem 1.2.B. *There exists a subset $P\subset(0;+\infty)$ of full Lebesgue measure such that for every $\tau\in P$ every trajectory of the kicked system ${f_*}^\tau$ is uniformly distributed in S^1.*

In general, the set P depends on the choice of kicks. Theorem 1.2.B is a consequence of the Weyl criterion. We prove it in a more general context of linear flows on tori in §2 below.

Example 1.2.C (Stable super-recurrence.) Let (X,Ω) be a closed symplectic manifold. Denote by μ the canonical probability measure on X. Let $G=\mathrm{Ham}(X,\Omega)$ be the group

of all Hamiltonian diffeomorphisms of (X, Ω). Recall that a symplectic diffeomorphism is called Hamiltonian if it can be included into a time-dependent Hamiltonian flow. A Hamiltonian function $H : X \times [0;1] \to \mathbb{R}$ is called **normalized** if $\int_X H(x,t)d\mu(x) = 0$ for all $t \in [0;1]$. Every Hamiltonian diffeomorphism $h \in G$ can be written as $h = h^1$, where h^t is the Hamiltonian flow generated by some normalized Hamiltonian H. In this case we say that h is generated by H. Define a function $\bar{\rho} : G \to [0; +\infty)$ as follows:

$$\bar{\rho}(h) = \inf \int_0^1 \max_{x \in X} H(x,t) - \min_{x \in X} H(x,t)dt,$$

where the infimum is taken over all normalized Hamiltonians H which generate h. The function $\bar{\rho}$ is known as *Hofer's norm* (see [H],[P2],[P4]). Let us introduce the following important notion. A sequence $\{\phi_i\} \in G^\infty$ is called **bounded** if the sequence $\{\bar{\rho}(\phi_i)\}$ is bounded.

We illustrate stable super-recurrence in the simplest case when $X = S^2$ is the unit sphere in $\mathbb{R}^3$ endowed with the induced Euclidean area form. In this case every diffeomorphism which preserves μ and the orientation is Hamiltonian. Every one-parameter subgroup (h^t) of G is simply an autonomous Hamiltonian flow generated by a (uniquely defined) time-independent Hamiltonian function $H \in \mathcal{H}$. Clearly the function H is an integral of motion (the energy conservation law!), and in particular each subset

$$A_\epsilon = \{H > (1-\epsilon)\max H\},\ \epsilon \in (0;1)$$

is invariant under the flow, and thus super-recurrent in the sense of Definition 1.1.D. Take an arbitrary bounded sequence of kicks $\{\phi_i\} \in G^\infty$. [2] Consider the kicked system $f_*{}^\tau = \{\phi_i h^\tau\}$. Fix $\epsilon \in (0;1)$.

Theorem 1.2.D. *Suppose that the Hamiltonian H is non-constant, and its maximum set $\{H = \max H\}$ contains a simple closed curve which divides the sphere into two discs of equal areas. Then there exists a subset $P \subset (0; +\infty)$ whose complement has a finite measure and such that for every $\tau \in P$*

- *the Hamiltonian H is a quasi-integral of the kicked system $f_*{}^\tau$;*
- *the set A_ϵ is super-recurrent for $f_*{}^\tau$.*

For instance, consider the flow (h^t) which rotates each point $(x,y,z) \in S^2$ with the velocity z around the z-axis. In Euclidean coordinates (x,y,z) on $\mathbb{R}^3$ it is given by

$$h^t(x,y,z) = (x\cos(2\pi tz) - y\sin(2\pi tz), x\sin(2\pi tz) + y\cos(2\pi tz), z). \tag{1.2.E}$$

It is generated by the Hamiltonian function $H(x,y,z) = -z^2 + \frac{1}{3}$ whose maximum set coincides with the equator $\{z = 0\}$. Therefore the theorem above is applicable, and in particular each annulus $\{|z| < \varepsilon\}$ is super-recurrent for the kicked system f_*^τ for all values of the period $\tau \in P$.

[2] For instance, one can think that all ϕ_i's are conjugate to elements of some compact (in the C^∞-topology) subset of G.

Theorem 1.2.D is proved in 4.3 below. Even in this simple case the proof we have is not elementary - it is based on powerful methods of modern symplectic topology. [3] We refer to section 1.6 for various extensions of this theorem to more general symplectic manifolds.

Example 1.2.F. (stable mixing) Let $\Gamma \subset PSL(2,\mathbb{R})$ be a lattice, that is a discrete subgroup such that the Haar measure μ of the quotient space $X = PSL(2,\mathbb{R})/\Gamma$ is finite. Let $G \subset PSL(2,\mathbb{R})$ be a *discrete* subgroup, and consider the left action of G on X. Let $h \in G$ be an element of infinite order. The Howe-Moore theorem [Zi] (see also 1.5.A below) yields that h is a mixing transformation of X. Let $\phi_* = \{\phi_i\}$ be an arbitrary sequence from G^∞ which represents a finite number of conjugacy classes in G. [4] Consider the kicked system ${f_*}^\tau = \{\phi_i h^\tau\}$, where $\tau \in \mathbb{N}$. We say that h is *stably mixing* if for every sequence ϕ_* as above there exists $\tau_0 > 0$ such that the kicked system ${f_*}^\tau$ is mixing for all $\tau > \tau_0$. The next result gives a complete description of stably mixing elements of G in purely algebraic terms.

Theorem 1.2.G. *Let $h \in G$ be an element of infinite order. The following conditions are equivalent:*

(i) h is stably mixing on X;

(ii) h is not conjugate to its inverse h^{-1} in G.

For instance, it is easy to see that the time one map of the horocycle flow on X which is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not conjugate to its inverse already in $PSL(2,\mathbb{R})$ and thus is stably mixing. On the other hand in $G = PSL(2,\mathbb{Z})$ every symmetric matrix is conjugate to its inverse by the involution

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and thus is not stably mixing. A complete description of $PSL(2,\mathbb{Z})$-matrices which are conjugate to their inverses is unknown (cf. [BR]). We refer to 1.7 for the fairly general discussion on the effect of time-reversing symmetry on the kick stability. The proof of Theorem 1.2.G is given in 1.5, 1.7 and §3 below. [5]

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[3] Dima Burago pointed out that in the case when the sequence of kicks $\{\phi_i\}$ is contained in a compact subset of G, the conclusions of 1.2.D can be checked by soft methods (compare with the previous footnote).

[4] For instance, if $\Gamma = PSL(2,\mathbb{Z})$ this assumption holds when all ϕ_i's are non-parabolic and their traces are bounded.

[5] The condition which tells that the sequence of kicks $\{\phi_i\}$ represents a finite number of conjugacy classes in G is essential as the next simple example shows. Let $G = PSL(2,\mathbb{Z})$ and $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Choose a surjective function $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\alpha^{-1}(\tau)$ is an infinite subset for every $\tau \in \mathbb{N}$. Put $q_k = k\alpha(k) - (k-1)\alpha(k-1)$, where $k \in \mathbb{N}$. Consider a sequence of kicks $\phi_k = h^{-q_k}$. Clearly it represents an infinite number of different conjugacy classes in $PSL(2,\mathbb{Z})$. The evolution of the kicked system is given by $f^{(k)}(\tau) = h^{k(\tau-\alpha(k))}$. Thus for every $\tau \in \mathbb{N}$ equality $f^{(k)}(\tau) = \mathbb{1}$ holds for an infinite number of k's, so the kicked system is not mixing.

1.3 Kick stability.

Now we are ready to give a formal definition of kick stability which provides a unified framework to examples considered above. In what follows $\mathcal{T}$ stands either for $\mathbb{R}$ or for $\mathbb{Z}$, and $\mathcal{T}_+ = \{t \in \mathcal{T} \mid t > 0\}$. We fix a class $\mathcal{B}$ of subsets of $\mathcal{T}_+$ of "large measure". For instance we can assume that when $\mathcal{T} = \mathbb{R}$ (resp. $\mathcal{T} = \mathbb{Z}$) the class $\mathcal{B}$ consists of all subsets of $\mathcal{T}_+$ whose complement has finite Lebesgue measure (resp. is a finite subset).

Let G be a group, and let (h^t), $t \in \mathcal{T}$ be a subgroup which is assumed to be either one parameter ($\mathcal{T} = \mathbb{R}$) or cyclic ($\mathcal{T} = \mathbb{Z}$). This subgroup represents the unperturbed dynamical system with continuous or discrete time. Fix a subset $\mathcal{P} \subset G^\infty$ which should be thought of as the set of all sequential systems with a given property (P). We say that our subgroup (h^t) has property (P) if the set

$$\{\tau \in \mathcal{T}_+ \mid \text{the sequence } \{h^\tau\} \in \mathcal{P}\}$$

has large measure, that is belongs to $\mathcal{B}$.

Let $\Phi \subset G^\infty$ be a set of admissible kicks. Take $\phi_* \in \Phi$ and write ${f_*}^\tau(\phi_*)$ for the kicked system $\{\phi_i h^\tau\}$. Denote by $P(\phi_*)$ the set of all positive values of the period τ such that the kicked system has property (P):

$$P(\phi_*) = \{\tau \in \mathcal{T}_+ \mid {f_*}^\tau(\phi_*) \in \mathcal{P}\}.$$

Formal Definition 1.3.A. The property (P) of subgroup (h^t) is *kick stable* if for every admissible sequence of kicks $\phi_* \in \Phi$ the set $P(\phi_*)$ of "good" periods belongs to the class $\mathcal{B}$ of subsets of large measure.

Let us illustrate this definition. In Example 1.2.A above, all kicks are admissible so the set Φ coincides with $(S^1)^\infty$. The set $\mathcal{P}$ consists of all sequences from $(S^1)^\infty$ whose orbits are uniformly distributed in S^1. Theorem 1.2.B implies that the property *all orbits are uniformly distributed in* S^1 is kick stable for (h^t). In Example 1.2.C the set Φ of admissible kicks consists of all sequences which are bounded in Hofer's norm. The set $\mathcal{P}$ is formed by all sequential systems for which the set A_ϵ is super-recurrent. Theorem 1.2.D states that the property A_ϵ *is a super-recurrent set* is kick stable for (h^t). In Example 1.2.F all sequences of kicks which represent a finite number of conjugacy classes are admissible, and the set $\mathcal{P}$ consists of all mixing systems. Theorem 1.2.G tells us when *mixing* is a kick stable property of the cyclic subgroup (h^t).

1.4 Sub-additive functions

We do not know of an argument which provides a unified explanation of the kick-stability phenomenon in all the examples presented in section 1.2 above. Interestingly enough, however that our approaches to Theorems 1.2.D and 1.2.G have a common ingredient. Namely, the desired kick stability is closely related to the geometric behaviour of the corresponding subgroups at infinity.

Definition 1.4.A. Let G be a group. A function $\rho : G \to [0; +\infty)$ is called *sub-additive* if there exists a number $C \geq 0$ such that the following holds:

- $|\rho(hgh^{-1}) - \rho(g)| \le C$ for all $g, h \in G$;
- $\rho(gh) \le \rho(g) + \rho(h) + C$ for all $g, h \in G$.

If ρ is sub-additive then, as is well known, for every $h \in G$, the limit

$$\rho_\infty(h) = \lim_{n\to+\infty} \frac{1}{n}\rho_\infty(h^n)$$

does exist. In the next two sections we discuss two applications of sub-additive functions on groups to kick stability.

1.5 Stable mixing and quasi-morphisms

In this section we outline the proof of the sufficient condition of stable mixing for an element of the discrete group G given in 1.2.G above.

We start with a following more general situation. Suppose that D is a non-compact simple Lie group with finite center. Let $\Gamma \subset D$ be a lattice, that is a discrete subgroup such that the Haar measure of the quotient space $X = D/\Gamma$ is finite. The group D acts on X on the left by transformations preserving the Haar measure. The key ingredient of our approach to stable mixing is the Howe-Moore theorem which provides a link between geometry and dynamics of sequential systems in this setting. In order to formulate it we need the following notions. Let $\{f^{(i)}\}$ be a sequence of elements of D. We say that $\{f^{(i)}\}$ *goes to infinity* if for every compact subset $Q \subset D$ there exists i_0 such that $f^{(i)} \notin Q$ for all $i > i_0$. From general considerations, it follows that a mixing sequence of elements of D necessarily goes to infinity [6]. Conversely, we have:

Howe-Moore Theorem 1.5.A. *([Zi]). Let f_* be a sequential system from D^∞. If its evolution $\{f^{(i)}\}$ goes to infinity then f_* is mixing.*

Let $G \subset D$ is a discrete group. Below we focus on the action of G on X. Consider the cyclic subgroup generated by an element $h \in G$. Let ϕ_* be an arbitrary sequence from G^∞ whose entries represent a finite number of conjugacy classes in G. Consider the kicked system ${f_*}^\tau = \{\phi_i h^\tau\}$, where $\tau \in \mathbb{N}$. The next result provides a sufficient condition for kick stability of mixing for our subgroup.

[6] Indeed, otherwise there is a subsequence which is *bounded* and therefore there is a *convergent* subsequence, which is also mixing. Thus (after passing to this subsequence) we have a sequence $f^{(i)} \to f_\infty$ and that is still mixing. Now take a real valued function G, not identically zero, with $\int_X G(x)d\mu(x) = 0$ and set $F(x) := G(f_\infty^{-1}x)$. By the mixing property of the sequence, we have

$$\int_X F(f^{(i)}x)G(x)d\mu(x) \to \int_X F(x)d\mu(x)\int_X G(x)d\mu(x) = 0$$

while because $f^{(i)} \to f_\infty$, we have

$$\int_X F(f^{(i)}x)G(x)d\mu(x) \to \int_X F(f_\infty x)G(x)d\mu(x) = \int_X G(x)^2 d\mu(x) > 0$$

which gives a contradiction.

Theorem 1.5.B. *Assume that there exists a sub-additive function ρ on G such that $\rho_\infty(h) > 0$. Then there exists $\tau_0 > 0$ such that the kicked system ${f_*}^\tau$ is mixing for all $\tau > \tau_0$.*

Here τ_0 depends on ϕ_*. It turns out that the geometric assumption $\rho_\infty(h) > 0$ guarantees that for large periods τ the evolution of the kicked system goes to infinity , thus the statement follows from 1.5.A. The details of this argument are given in 3.1 below.

There exists a useful class of sub-additive functions which arise naturally in the bounded cohomology theory of discrete groups (see [Br],[BG],[Pi]).

Definition 1.5.C. A function $r : G \to \mathbb{R}$ is called a *quasi-morphism* if there exists a constant $C > 0$ such that

$$|r(gh) - r(g) - r(h)| \leq C$$

for all $g, h \in G$.

Given a quasi-morphism r and an element $g \in G$, there exists the limit

$$r_\infty(g) = \lim_{n \to +\infty} \frac{r(g^n)}{n}.$$

Note that r_∞ is homogeneous, that is $r_\infty(g^k) = kr_\infty(g)$. Moreover, if g has finite order then $r_\infty(g) = 0$. It follows immediately from the definition that if r is a quasi-morphism then the function $\rho(g) = |r(g)|$ is sub-additive, and moreover $\rho_\infty(g) = |r_\infty(g)|$ for all $g \in G$.

Theorem 1.5.D. *Let $G \subset PSL(2, \mathbb{R})$ be a discrete group and h an element of infinite order in G. The following conditions are equivalent:*

(i) there exists a quasi-morphism $r : G \to \mathbb{R}$ such that $r_\infty(h) > 0$;

(ii) h is not conjugate to its inverse h^{-1} in G.

Proof of "1.2.G(ii) implies 1.2.G(i)": The desired statement is an immediate consequence of 1.5.D and 1.5.B. □

Theorem 1.5.D is proved in 3.2 below (see Remark 3.2.F for references and generalizations of this result).

1.6 Stable super-recurrence in Hamiltonian dynamics

Let (X, Ω) be a closed symplectic manifold. Denote by μ the canonical probability measure on X. Let $G = \mathrm{Ham}(X, \Omega)$ be the group of all Hamiltonian diffeomorphisms of (X, Ω). Define *the positive part of Hofer's norm* $\rho : G \to [0; +\infty)$ as follows:

$$\rho(h) = \inf \int_0^1 \max_{x \in X} H(x, t) dt,$$

where the infimum is taken over all normalized Hamiltonian functions $H : X \times [0; 1] \to \mathbb{R}$ which generate h (cf. 1.2.C above). It is an easy exercise to check that ρ is sub-additive

(here the constant C of 1.4.A is simply 0). Moreover, the following obvious inequality holds:$\bar{\rho}(h) \geq \rho(h) + \rho(h^{-1})$ for all $h \in G$. [7]

Let (h^t) be a one-parameter subgroup of G. It is generated by some uniquely defined time-independent Hamiltonian $H : X \to \mathbb{R}$ with zero mean. The law of energy conservation yields that H is an integral of motion. Note that $\rho(h^t) \leq t \max H$ for all $t > 0$, and thus we have $\rho_\infty(h^1) \leq \max H$. Let $\phi_* \in G^\infty$ be an arbitrary bounded sequence of kicks (see 1.2.C above). Consider the kicked system ${f_*}^\tau = \{\phi_i g^\tau\}$.

Theorem 1.6.A. *(stable energy conservation law). Assume that $\rho_\infty(h^1) > 0$. There exists a subset $P \subset (0; +\infty)$ of density at least*

$$\frac{\rho_\infty(h^1)}{\max H}$$

such that for every $\tau \in P$ the Hamiltonian H is a quasi-integral of the kicked system ${f_*}^\tau$.

This result gives a probabilistic interpretation for a purely geometric quantity $\rho_\infty(h^1)$. Further, this quantity contains interesting information about kick stable super-recurrence of (h^t)-invariant sets of the form

$$A_\epsilon = \{H > \epsilon \max H\}.$$

We present here two sample results.

Theorem 1.6.B. *Assume that $\rho_\infty(h^1) = \max H$. Fix $\epsilon > 0$. There exists a subset $P \subset (0; +\infty)$ of density 1 such that for every $\tau \in P$ the set A_ϵ is super-recurrent for the kicked system ${f_*}^\tau$.*

One can check (see 4.2, 4.3 below) that in Example 1.2.C above the assumption $\rho_\infty(h^1) = \max H$ is satisfied, thus (a slightly weaker version of) Theorem 1.2.D follows from 1.6.B.

Theorem 1.6.C. *Assume that $\rho_\infty(h^1) \geq 0.9 \max H$, and $\max H = -\min H$. Then there exists a subset $P \subset (0; +\infty)$ of density at least 0.4 such that for every $\tau \in P$ the set $A_{0.4}$ is super-recurrent for the kicked system ${f_*}^\tau$.*

We prove more general versions of these theorems in §4 below.

1.7 An obstruction to kick stability

We describe here a method of constructing kick unstable systems in a number of interesting situations. Let (h^t), $t \in \mathcal{T}$ be a one-parameter/cyclic subgroup of a group G.

Definition 1.7.A. An element $\theta \in G$ is called *a time-reversing symmetry* [8] for (h^t) if $\theta h^t \theta^{-1} = h^{-t}$ for all $t \in \mathcal{T}$. Let us introduce the following notation. Given a system $f_* \in G^\infty$ we write $f^* = \{f^{(i)}\}$ for its evolution $f^{(i)} = f_i ... f_1$.

[7] In fact, in all known examples one has the equality!

[8] See [LR] for a discussion on time-reversing symmetries and their impact on dynamics.

1.7.B. Creating periodic behaviour. Suppose that (h^t) admits a time-reversing symmetry θ. Take a sequence of kicks

$$\phi_* = \{\theta^{-1}, \theta, \theta^{-1}, \theta, ...\}.$$

Consider the kicked system $f_*(\tau) = \{\phi_i h^\tau\}$. Its evolution $f^*(\tau)$ is 2-periodic:

$$f^*(\tau) = \{\theta^{-1}h^\tau, \mathbb{1}, \theta^{-1}h^\tau, \mathbb{1}, ...\}.$$

Assume that G acts by measure-preserving homeomorphisms on a topological space X. Obviously, for all $\tau \in \mathcal{T}$ every subset A of X with $\mu(A) < 0.5$ is super-recurrent for the kicked system $f_*(\tau)$.

Example 1.7.C. (cf. 1.2.A above). Consider the action of the orthogonal group $O(2)$ on the circle S^1. The uniform distribution property of the flow $h^t x = x + t$ is not kick stable for kicks from $(O(2))^\infty$. Indeed, the transformation $\theta : x \to -x$ is a time-reversing symmetry for h^t. Comparing this with Theorem 1.2.B above we see that the flow h^t loses stability when one replaces the group $S^1 = SO(2)$ by a larger group $O(2)$. This stability breaking mechanism can be observed in many other situations (see 1.7.H and 4.6.A below).

1.7.D. Suppose now in addition that every point $x \in X$ admits a nested system of open neighborhoods U_δ, $\delta \in (0; \delta_0)$ such that $\cap_\delta U_\delta = \{x\}$ and $\mu(U_\delta) = \delta$. We claim that for every τ the kicked system $f_*(\tau)$ constructed above is not mixing. Indeed, fix arbitrary $\tau \in \mathcal{T}$, and take $\delta > 0$ small enough. Since $\theta \neq h^\tau$ there exists an open subset U of X of measure δ such that $\theta^{-1}h^\tau U \cap U = \emptyset$. Let F be a characteristic function of U. The only limit points of the sequence $\int_X F(f^{(i)}(\tau)x)F(x)d\mu(x)$ are 0 and δ, while $\left(\int_X F(x)d\mu(x)\right)^2 = \delta^2$. Therefore the kicked system is not mixing.

Proof of "1.2.G(i) implies 1.2.G(ii)": Consider the action of G on $X = PSL(2,\mathbb{R})/\Gamma$. Let $h \in G$ be an element of infinite order. If h is conjugate to its inverse (that is it admits a time-reversing symmetry) then the argument 1.7.D above shows that h is not stably mixing. This completes the proof of 1.2.G. □

1.7.E. Creating random behaviour. Exactly as in examples above, a time-reversing symmetry provides an obstruction to stable super-recurrence in Hamiltonian dynamics. We work in the setting of 1.6, assuming in addition that the group of Hamiltonian diffeomorphisms G is C^∞-closed in the group of all smooth diffeomorphisms of X. [9] The next result is proved in 4.6 below.

Proposition 1.7.F. *Assume that (h^t) is a one-parameter subgroup of the group G of Hamiltonian diffeomorphisms of X which admits a time-reversing symmetry. Then there exists a bounded sequence $\phi_* = \{\phi_i\} \in G^\infty$ such that the kicked system $\{\phi_i h^\tau\}$ is strictly ergodic for all $\tau \in (0; +\infty)$.*

Example 1.7.G (cf. 1.2.C above.) Let X be the 2-sphere, and let h^t be the circle action which rotates the sphere with constant speed around the z-axes. This flow has no

[9] This is true for many symplectic manifolds, for instance if $H^1(X,\mathbb{R}) = 0$ or if the cohomology class of the symplectic form Ω is rational. The famous Flux Conjecture (see e.g. [LMP]) which states that this is true for all (X,Ω), is still open.

stably super-recurrent invariant sets with piece-wise smooth boundary. Indeed, it has a time-reversing symmetry (for instance, the reflection around the x-axes).

Example 1.7.H (cf. 1.2.C above.) Assume now that h^t is the flow given by (1.2.E) above. Instead of the group of all Hamiltonian diffeomorphisms of S^2 consider the larger group of all measure-preserving diffeomorphisms [10]. It turns out that after such an enlarging of the ambient group, the flow (h^t) gets a time-reversing symmetry- the reflection over the (x, y)-plane. A little modification of 1.7.F above yields that (h^t) loses stable super-recurrence (cf. 1.7.C above).

1.8. Discussion and open problems

1.8.A. Historical and bibliographical remarks.

Sequential systems arise naturally in random dynamics [Ki]. In the deterministic language they form a particular case of a skew-product. The study of ergodic properties of individual sequential systems was initiated by Bergelson and Berend [BB]. Some special classes of sequential systems were known for a long time. For instance, sequences of Möbius transformations were considered in connection with analytic continued fractions as well as with the discrete Schrödinger operator. Very recently Zeghib [Ze] investigated sequences of isometries of a Lorentz manifold. He found that the corresponding dynamics is closely related to asymptotic geometry of the isometries group (cf. 1.5-1.6 above). It is clear that sequential dynamics have not been systematically studied yet, and many natural and interesting questions are still unexplored.

Kicked systems were intensively studied by physicists in the classical (= non-sequential) framework. This class of systems includes a number of famous maps which attracted a lot of attention in conservative chaotic dynamics, such as the kicked top [HKS], the kicked harmonic oscillator (or the Henon map [He]) and the kicked rotator (or the standard map [Ha]). Physicists, however considered these maps from a viewpoint which essentially differs from ours. In order to illustrate the difference, let us return to the the flow (h^t) given by (1.2.E) above. Take the constant sequence of kicks $\phi_i \equiv \phi$, where $\phi(x, y, z) = (-z, y, x)$. The corresponding kicked system describes iterations of the single map ϕh^τ, which is nothing else but *the kicked top map* (see [HKS]). In contrast to our setting, in the physics literature ϕ describes the top, while h^τ stands for the kick! Further, computer experiments performed in [HKS] suggest that for large values of τ the kicked system is chaotic: a generic trajectory of the kicked top map is uniformly distributed on a huge subset of the sphere. On the other hand, Theorem 1.2.D above guarantees super-recurrence for all positive τ outside the set of finite measure. We arrive at a seemingly paradoxical situation: chaos coexists with super-recurrence. The resolution of this paradox is as follows. The deterministic behaviour of a *generic* trajectory of the unperturbed system is a kick unstable property. This kick instability, reflected in the transition to chaos for large values of τ, is the main attraction for physicists. The main message of our theory is that even for large τ some (non-generic!) trajectories are still super-recurrent. Indeed, our notion of super-recurrence takes into account behaviour of *all* (vs. almost-all) trajectories of a system. For instance, the original

[10] Besides the Hamiltonian ones, this group includes orientation reversing diffeomorphisms preserving the measure.

kicked top map is super-recurrent simply due to existence a fixed point $(1, 1, 0)$. It surprising however that this super-recurrence persists when the constant sequence of kicks is replaced by an arbitrary bounded non-constant sequence. Let us mention also that this conflict between "all" and "almost all" is a reflection of the striking difference between L^p- and C^0- measurements in symplectic topology, see for instance [EP],[P4]. A similar analysis of the standard map (which we omit here) also leads to some "paradoxical" conclusions which are far from being understood yet.

Kick stability, the central notion promoted in the present paper, lies in a long series of attempts to formalize robustness of dynamical and ergodic properties of flows and maps (such as recent works on stable ergodicity, see [FP] and references therein). Among its cousins one may recall stochastic stability [V] which naively speaking means stability with respect to small random sequential kicks. It would be interesting to compare kick stability (where the kicks are deterministic and not assumed to be small) with stochastic stability in more detail.

1.8.B. Discrete vs. continuous

In basic examples considered above- uniform distribution on S^1, super-recurrence in Hamiltonian dynamics and mixing on $PSL(2,\mathbb{R})/\Gamma$ - one can address both discrete and continuous versions of the question on kick stability. In the present paper we worked out the continuous versions in the first two examples, and the discrete version in the last one. What happens with the remaining cases? It turns out that cyclic subgroups of S^1 do not have kick stable uniform distribution property. This was noticed by Dima Burago whose argument is presented in 2.3 below. Further, nothing is known to us about kick stable super-recurrence for cyclic subgroups of the group of Hamiltonian diffeomorphisms of a symplectic manifold. It would be interesting to make some progress in this direction. Finally, we arrive at the kick stability question for flows on $PSL(2,\mathbb{R})/\Gamma$. It deserves a special discussion.

Consider the action of $G = PSL(2,\mathbb{R})$ on $X = PSL(2,\mathbb{R})/\Gamma$, where Γ is a lattice. It follows from 1.5.A that every non-compact one-parameter subgroup of G is mixing. Is this property kick stable? For instance, the geodesic flow

$$h^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is not stably mixing. Indeed, it is given by a symmetric matrix and thus admits a time-reversing symmetry (see discussion in 1.7.D above).

Question 1.8.C. Is the horocycle flow

$$h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

stably mixing?

Note that the horocycle flow does not admit a time reversing symmetry in $PSL(2,\mathbb{R})$. Being unable to find the complete answer, we present some partial results (suggesting the affirmative solution) and more discussion in the Appendix to §3 below. In particular, we present a link between this problem and spectral theory for the discrete Shrödinger equation.

1.8.D. Generalizations to other dynamical systems.

It would be interesting to investigate stable mixing for lattices in semi-simple Lie groups of higher rank. Does there exists a complete description of stably mixing elements (if any) similar to Theorem 1.2.G above? Recently Burger and Monod [BM1, BM2] showed that unlike the rank-one case, there are few quasi-morphisms of lattices in higher rank groups. Thus new ideas are needed in order to understand that case.

Another potential source of kick-stable systems might be provided by hyperbolic theory. Here is a warm up question: under which conditions is the existence of a hyperbolic attractor a kick stable property?

CONTENTS

Acknowledgments: We thank Dima Burago for a number of important critical remarks and suggestions, and in particular for explaining us a construction presented in 2.3 below. We are grateful to Anna Dioubina for pointing out an error in the original version of Remark 3.3.E, as well as to Brian Bowditch and Iosif Polterovich for illuminating consultations on reference [EF]. We thank Leonid Pastur and Misha Sodin for various useful discussions.

2 Linear flows on tori

2.1. Generic flows are kick stable.

In this section we consider kick-stability for linear flows on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. It turns out that some very classical results on the "metric" theory of uniform distribution immediately imply that the kicked flows are stably uniformly distributed for almost all periods.

Precisely, for $\omega \in \mathbb{R}^d$ consider the "Kronecker map" on the torus $\mathbb{T}^d$ given by $x \mapsto x + \omega \mod 1$. It gives a one-parameter group ($t \in \mathbb{R}$)

$$h^t(x) = x + t\omega \mod 1 .$$

Given $\beta_i \in \mathbb{T}^d$, we get "kicks" $\phi_i(x) = x + \beta_i \mod 1$. Put $f_i^\tau = \phi_i h^\tau$, so that $f_i^\tau(x) = x + \beta_i + \tau\omega \mod 1$. The evolution of the kicked system is then given by

$$f^{(k)} = f_k^\tau f_{k-1}^\tau \dots f_1^\tau : x \mapsto x + \alpha_k + k\tau\omega \mod 1$$

where $\alpha_k = \beta_1 + \cdots + \beta_k$.

We will say that $\omega = (\omega_1, \dots, \omega_d)$ is a "generic" vector if its components ω_i are linearly independent over the rationals.

Theorem 2.1.A. *Suppose that ω is a "generic" vector. Then for almost all periods τ, the orbits $\{f^{(k)}(x)\}_{k=1}^\infty$ are uniformly distributed in $\mathbb{T}^d$.*

This result is a version of a "metric theorem" of Weyl from 1916 (see [KN]). For the sake of completeness we recall the argument. In the following, given two sequences $a(N)$ and $b(N)$ we will use the notation $a(N) \ll b(N)$ to mean that there is a constant $c > 0$ so that $a(N) \leq cb(N)$ for all N sufficiently large.

2.2. Proof of 2.1.A:

Since $\mathbb{T}^d$ acts transitively on itself by translations, it suffices to consider the base point 0 instead of arbitrary x. It also suffices to fix a finite interval $[a, b]$ and show that the result holds for almost all $\tau \in [a, b]$.

Define normalized "Weyl sums"

$$S_h(N, \tau) := \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i h \cdot f^{(k)}(0)}$$

The basic tool is Weyl's observation that for uniform distribution, it suffices to show that the normalized Weyl sums converge to zero for all integer vectors $h \neq 0$.

To do that, one shows (see below) that for fixed $h \neq 0$, one has

$$(2.2.A) \qquad \int_a^b |S_h(N, \tau)|^2 \, d\tau \ll \frac{\log N}{N}.$$

Thus for the sequence of squares $N = n^2$, we have

$$\sum_{n=1}^{\infty} \int_a^b \left|S_h(n^2, \tau)\right|^2 d\tau < \infty \, .$$

By Fatou's lemma, it follows that $\sum_{n=1}^{\infty} \left|S_h(n^2, \tau)\right|^2$ is integrable on $[a, b]$, and so is finite for all τ in a set of full Lebesgue measure P_h. Thus the n-th term tends to zero for all $\tau \in P_h$. Intersecting over all integer vectors $h \neq 0$ we get one set P of full measure which works for all $h \neq 0$, that is for all $\tau \in P$

$$S_h(n^2, \tau) \to 0, \qquad n \to \infty \, .$$

Now given any N, find n so that $n^2 \leq N < (n + 1)^2$. Writing $N = n^2 + k$, $0 \leq k \leq 2n$ we have by using the trivial bound $|e^{2\pi i x}| \leq 1$ that

$$\left|S_h(n^2 + k, \tau) - S_h(n^2, \tau)\right| \ll \frac{k}{n^2} \ll \frac{1}{n} \to 0, \qquad n \to \infty$$

for $\tau \in P$ and since $S_h(n^2, \tau) \to 0$, we get $S_h(N, \tau) \to 0$ for all $N \to \infty$.

To show 2.2.A, we square out the sum and directly integrate to get

$$\int_a^b |S_h(N, \tau)|^2 \, d\tau = \frac{1}{N^2} \sum_{k,l \leq N} e^{2\pi i h \cdot (\alpha_k - \alpha_l)} \int_a^b e^{2\pi i (k-l) h \cdot \omega \tau} d\tau \, .$$

The "diagonal" terms $k = l$ give a total contribution of $(b - a)/N$ to the sum, so to prove 2.2.A it suffices to bound the off-diagonal terms $k \neq l$.

Since $\omega \cdot h \neq 0$ for integer $h \neq 0$ (that was the assumption on ω), each off-diagonal term contributes

$$e^{2\pi i h \cdot (\alpha_k - \alpha_l)} \frac{e^{2\pi i (k-l) h \cdot \omega b} - e^{2\pi i (k-l) h \cdot \omega a}}{2\pi i (k - l) h \cdot \omega} \, .$$

Taking absolute values and summing over all pairs $1 \leq k \neq l \leq N$ gives a contribution bounded by a constant times

$$(2.2.B) \qquad \frac{1}{N^2} \sum_{1 \leq k \neq l \leq N} \frac{1}{|k - l|}.$$

For fixed $n \neq 0$, the number of solution of $k - l = n$ with $1 \leq k \neq l \leq N$ is $N - |n|$ if $1 \leq |n| \leq N - 1$ and zero otherwise. Thus 2.2.B is given by

$$\frac{2}{N^2} \sum_{n=1}^{N-1} \frac{N - n}{n} \ll \frac{\log N}{N}$$

which proves 2.2.A and the Theorem. □

2.3 Cyclic subgroups of S^1 are kick unstable.

We present here a counter-example to kick stability constructed by Dima Burago. In what follows we identify Kronecker maps $x \to x + b$ with the corresponding elements $b \in S^1$. Fix

an irrational element $\omega \in S^1$. The corresponding cyclic subgroup is uniformly distributed in S^1. We claim that there exists a sequence of kicks $\{\beta_i\} \in (S^1)^\infty$ such that for every $\tau \in \mathbb{N}$ the evolution of the kicked system is **not** uniformly distributed in S^1. Recall from 2.1 that this evolution is given by $f^{(k)} = \alpha_k + k\tau\omega \mod 1$, where $\alpha_k = \beta_1 + ... + \beta_k$. In order to prove the claim, choose a function $u : \mathbb{N} \to \mathbb{N}$ such that the preimage $u^{-1}(k) \subset \mathbb{N}$ of every integer $k \in N$ is a subset of strictly positive density. Put now $\alpha_k = -u(k)k\omega \mod 1$, and $\beta_k = \alpha_k - \alpha_{k-1}$. Fix $\tau \in \mathbb{N}$ and consider the sequence $\{f^{(k)}\}$. Since every element of this sequence with $k \in u^{-1}(\tau)$ vanishes, and the set $u^{-1}(k)$ has positive density in $\mathbb{N}$, we conclude that this sequence is not uniformly distributed in S^1. This completes the proof of the claim.

3. Detecting stable mixing on $PSL(2,\mathbb{R})/\Gamma$

In this section we prove Theorems 1.5.B,D and in an Appendix, present some partial answers on Question 1.8.C.

3.1. Proof of 1.5.B:

Let C be the constant from Definition 1.4.A. Put $C_1 = \max_{j\in\mathbb{N}} \rho(\phi_j^{-1})$. The maximum is finite since $\{\phi_j\}$ represents a finite number of conjugacy classes, and ρ is bi-invariant up to C (see 1.4.A). Denote by $f^{(k)}(\tau)$ the evolution of the kicked system,

$$f^{(k)}(\tau) = \phi_k h^\tau \dots \phi_1 h^\tau \ .$$

Note that for every $k > 0$ and $\tau \in (0 + \infty)$

$$h^{\tau k} = f^{(k)}(\tau) \cdot \prod_{j=1}^{k} h^{-\tau j} \phi_j^{-1} h^{\tau j} \ .$$

Applying ρ to both sides of this equation and using properties listed in 1.4.A we get that

$$\rho(h^{\tau k}) \le \rho(f^{(k)}(\tau)) + \sum_{j=1}^{k} \rho(\phi_j^{-1}) + 2Ck \tag{3.1.A}$$

$$\le \rho(f^{(k)}(\tau)) + k(2C + C_1) \ .$$

Choose $\tau_0 > 0$ so large that $\rho(h^{\tau k}) \ge 0.5\tau k \rho_\infty(h)$ for all $\tau > \tau_0$ and $k \in \mathbb{N}$. Put $C_2 = 0.5\tau_0\rho_\infty(h) - (2C + C_1)$. Increasing if necessary τ_0 we assume that $C_2 > 0$. In view of (3.1.A) we have $\rho(f^{(k)}(\tau)) \ge C_2 k$ for $\tau > \tau_0$. Thus for $\tau \ge \tau_0$ the sequence $f^{(k)}(\tau)$ goes to infinity (see 1.5). Applying the Howe-Moore theorem 1.5.A we see that for $\tau > \tau_0$ the kicked system is mixing. □

3.2. Quasi-morphisms

Our purpose is to prove Theorem 1.5.D, that is if $G \subset PSL(2,\mathbb{R})$ is a discrete group, and $g \in G$ an element of infinite order, then the following are equivalent:

(i) There exists a quasi-morphism $r : G \to \mathbb{R}$ so that $r_\infty(g) \ne 0$.

(ii) g is not conjugate to its inverse g^{-1} (in which case we say that g does not admit a time-reversing symmetry in G).

One direction is immediate: Given a quasi-morphism r, we note that r_∞ is *homogeneous*: $r_\infty(g^k) = kr_\infty(g)$ for $k \in \mathbb{Z}$ and so if g has finite order then clearly $r_\infty(g) = 0$. Moreover, if $g = hg^{-1}h^{-1}$ with $h \in G$ then also $g^k = hg^{-k}h^{-1}$ for all $k \ge 1$. Since r_∞ is a homogeneous quasi-morphism, we have

$$r_\infty(g^k) = r_\infty(hg^{-k}h^{-1}) = r_\infty(g^{-k}) + O(1) = -kr_\infty(g) + O(1)$$

and consequently $2k|r_\infty(g)| = O(1)$ is bounded for all $k \ge 1$, which forces $r_\infty(g) = 0$.

Let us show now that (ii) yields (i). Thus if G is a discrete subgroup of $PSL(2,\mathbb{R})$, we wish to show that given any element of G of infinite order which is not conjugate in G to its inverse, there is a homogeneous quasi-morphism $r = r_\infty$ of G for which $r_\infty(g) \neq 0$.

Remark. Concerning the condition that g is conjugate in G to its inverse g^{-1}, we note that this can only happen for *hyperbolic* g since elliptic and parabolic elements of $PSL(2,\mathbb{R})$ are never conjugate to their inverses in $PSL(2,\mathbb{R})$. Moreover, if $g \in PSL(2,\mathbb{R})$ is hyperbolic then it can be shown that any time reversing symmetry K of g (i.e. an element K such that $g = Kg^{-1}K^{-1}$) must satisfy $K^2 = -1$ (in $SL(2,\mathbb{R})$). Thus for many cases of interest, such as surface groups, which do not have elliptic elements, this possibility does not arise.

We will use the following well-known construction (see [BG] and [Pi], 3.3.2). For a pair of points x, y in the hyperbolic plane $\mathbf{H}$, write $\ell(x,y)$ for the oriented geodesic joining x to y. Let $\Omega = dx \wedge dy/y^2$ be the hyperbolic area form.

Definition 3.2.A. *A one-form α on $\mathbf{H}$ is* bounded *if there is some $C > 0$ so that $|\frac{d\alpha}{\Omega}| \leq C$.*

Given a bounded G-invariant one-form on $\mathbf{H}$ and a base-point $x \in \mathbf{H}$, set

$$r_x(g) = r_x^\alpha(g) := \int_{\ell(x,gx)} \alpha$$

Lemma 3.2.B. *If α is a bounded G-invariant one-form on $\mathbf{H}$ then*

(i) r_x is a quasi-morphism.

(ii) $|r_x - r_y| \leq C_\alpha$.

Proof: 1) Let $g, h \in G$ and consider

$$\delta r_x(g,h) := r_x(g) + r_x(h) - r_x(gh) \ ,$$

which we want to show is bounded. By G-invariance of α, we have

$$r_x(h) := \int_{\ell(x,hx)} \alpha = \int_{\ell(gx,ghx)} \alpha$$

Therefore

$$\delta r_x(g,h) = \left(\int_{\ell(x,gx)} + \int_{\ell(gx,ghx)} - \int_{\ell(x,ghx)} \right) \alpha = \int_{\partial T} \alpha$$

is the integral around the oriented boundary of the geodesic triangle T with vertices at x, gx and ghx. By Stokes' theorem, this equals the integral of $d\alpha$ on T, and thus if $|d\alpha| \leq C\Omega$ then

$$|\delta r_x(g,h)| = \left| \int_T d\alpha \right| \leq C \left| \int_T \Omega \right| = C \cdot \text{area(T)} \ .$$

Since the area of a geodesic triangle in the hyperbolic plane is at most π, we find that $|\delta r_x(g,h)| \leq \pi C$ is bounded and thus r_x is a quasi-morphism.

2) To see independence of r_x on the base-point up to a bounded quantity, consider the integral of α over the boundary of the geodesic parallelogram P with vertices at x, gx, gy and y, and again use Stokes' theorem:

$$\left(\int_{\ell(x,gx)} + \int_{\ell(gx,gy)} + \int_{\ell(gy,y)} + \int_{\ell(y,x)} \right) \alpha = \int_P d\alpha = O(1) \ .$$

By G-invariance of α, we have

$$\int_{\ell(gx,gy)} \alpha = \int_{\ell(x,y)} \alpha = -\int_{\ell(y,x)} \alpha$$

and so we find

$$|r_x(g) - r_y(g)| = \left| \int_{\ell(x,gx)} + \int_{\ell(gy,y)} \right| = \left| \int_P d\alpha \right| \leq 2\pi C$$

since the hyperbolic area of P is at most 2π. □

Since G is discrete, there are two kinds of elements of G with infinite order: hyperbolic and parabolic. The construction of r_∞ is carried out separately for each of these two cases.

The hyperbolic case. It suffices to consider the case that g is a *primitive* hyperbolic element of G, that is we cannot write $g = g_1^k$ for some $g_1 \in G$ and $|k| \geq 2$. Thus we assume this to be the case from now.

We recall some facts from the geometry of discrete groups:

Any hyperbolic element leaves invariant a unique geodesic in $\mathbf{H}$. Let L be the invariant geodesic for a primitive hyperbolic element g. The following is a standard fact:

Lemma 3.2.C.

(i) Suppose $\gamma \in G$, $\gamma L = L$ and γ preserves the orientation of L. Then $\gamma = g^k$ for some integer k.

(ii) Suppose $\gamma \in G$, $\gamma L = L$ and γ reverses the orientation of L. Then $\gamma g \gamma^{-1} = g^{-1}$.

Any discrete group G admits a fundamental region D for which the tessellation $\{\gamma \overline{D} : \gamma \in G\}$ of the upper half-plane $\mathbf{H}$ is *locally finite*, that is to say each compact subset of $\mathbf{H}$ intersects only finitely many of the translates $\gamma\overline{D}$. An example is the Dirichlet fundamental region of G [Be].

Lemma 3.2.D. *There is a segment $\mathcal{I} \subset L$ and $\epsilon > 0$ such that for every $\gamma \in G$, either $\gamma\mathcal{I} \subset L$ or else $dist(\gamma\mathcal{I}, L) > \epsilon$.*

Proof: Let D be a locally finite fundamental domain of G which intersects L at an interior point. Write U_δ for the δ-neighborhood of L in the hyperbolic metric (the so-called "hypercycle domain"). Because D is locally finite, there are only a *finite* number of distinct translates γU_δ, $\gamma \in G$, which intersect the closure $\overline{D}$ of D (see e.g. [Be, Theorem 9.2.8 (iii)]).

Let Y be the union of these finitely many translates of U_δ, which are distinct from U_δ itself. Decreasing δ, we can guarantee that $Y \cap L \cap D \neq L \cap D$.

Choose a segment $\mathcal{I} \subset L \cap D$ such that $\mathcal{I} \cap Y = \emptyset$. Assume that for some $\gamma \in G$, $\gamma\mathcal{I} \cap U_\delta \neq \emptyset$. Then $\mathcal{I} \cap \gamma^{-1} U_\delta \neq \emptyset$. But this means that $\gamma^{-1} U_\delta = U_\delta$ due to our construction. This implies that $\gamma^{-1} L = L$ so that $\gamma\mathcal{I} \subset L$. This proves the lemma (with $\epsilon = \delta$). □

Proof of the theorem in the hyperbolic case: Assume that g does not admit a time-reversing symmetry. Then Lemmas 3.2.C and 3.2.D imply that if $\gamma \neq g^k$ for some $k \in \mathbb{Z}$ then $\gamma\mathcal{I}$ is bounded away from L. Shrinking if necessary the segment $\mathcal{I}$, we see that there exists its small neighborhood $\mathcal{U}$ such that $\gamma\mathcal{U}$ is bounded away from $\mathcal{U}$ for all $1 \neq \gamma \in G$.

Choose a one-form α_0 in $\mathbf{H}$ such that α_0 has compact support, contained in $\mathcal{U}$ and

$$\int_{\mathcal{I}} \alpha_0 > 0.$$

Let α be the extension of α_0 by periodicity to the translates $\cup_{\gamma\in G}\gamma\mathcal{U}$. Note that α is a *bounded* one-form.

Now fix a point $x \in L \cap D$ such that $\mathcal{U} \cap L$ lies in the interior of the geodesic segment $\ell(x, gx)$, and let $r = r_x^\alpha$ be the quasi-morphism constructed above. Clearly $r(g^n) = \int_{\ell(x,g^n x)} \alpha = n \int_{\mathcal{I}} \alpha_0$ and so $r_\infty(g) = \lim r(g^n)/n = \int_{\mathcal{I}} \alpha_0 > 0$. This completes the proof.

The parabolic case. Let $h \in G$ be a parabolic element, which as in the hyperbolic case we may assume is *primitive*. Recall that parabolic elements are never conjugate to their inverses already in $PSL(2, \mathbb{R})$. Let L be an h-invariant horocycle and let $\mathcal{U}_L$ be a horocyclic domain in $\mathbf{H}$, that is a neighborhood of the cusp fixed by h. For instance, if ∞ is a cusp for G and $h(z) = z+1$ and we can take $L = \{y = C\}$ and $\mathcal{U}_L = \{y \geq C\}$. The following is well-known:

Lemma 3.2.E. *One can choose L so that $\gamma\mathcal{U}_L \cap \mathcal{U}_L = \emptyset$ for all $\gamma \in G$ with $\gamma \neq h^k$.*

Proof: We may use a normal form for h and so assume that the cusp is at ∞ and that $h(z) = z + 1$. Then if $\gamma \in G$ does not fix the cusp ∞, then $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $|c| \geq 1$ (see e.g. [Be, Proof of 9.2.8 (ii)]). In that case $\mathrm{Im}(\gamma z) = \mathrm{Im}(z)/|cz + d|^2 \leq 1/y$. Thus the horocycles $L = \{y = C\}$ for $C > 1$ satisfy the conditions of the Lemma. $\square$

Proof of the theorem in the parabolic case: For simplicity we assume that $h(z) = z+1$, $z = x + iy$. Take $L = \{y = 2\}$ and let $u(y)$ be a smooth cutoff function, with $u(y) \equiv 1$ for $y \geq 3$, and $u(y) \equiv 0$ if $y \leq 2.5$. Set $\alpha_0 = u(y)dx$.

Note that α_0 is *bounded* on $\mathbf{H}$, since $d\alpha_0 \equiv 0$ for $y \notin (2.5, 3)$, while for $2.5 < y < 3$ we have $d\alpha_0 = u'(y)dy \wedge dx$ and comparing with $\Omega = y^{-2}dx \wedge dy$ gives $|d\alpha_0/\Omega| = y^2 u'(y)$ is bounded. Moreover α_0 is supported in $\mathcal{U}_L = \{y \geq 2\}$.

Now let $\alpha = \sum_{\gamma\in G/\langle h\rangle} \gamma^*\alpha_0$ be the periodization of α_0. The translates $\gamma^*\alpha_0$ are supported in distinct translates $\gamma\mathcal{U}_L$ for distinct γ modulo translates by powers of h. Thus we get a G-invariant, bounded one-form on $\mathbf{H}$ which equals dx on $\{y \geq 3\}$.

Choose z with $\mathrm{Im}(z) = 3$ and consider the quasi-morphism $r = r_z^\alpha$. Then $h^n(z) = z + n$ and clearly $\int_{\ell(z,h^n z)} \alpha = n$. Thus $r_\infty(h) = 1 > 0$ as required. This completes the proof of Theorem 1.5.D. $\square$

Remark 3.2.F. The phenomenon described in Theorem 1.5.D holds true in a more general context of Gromov hyperbolic groups. In fact, if G is a non-elementary Gromov hyperbolic group then for every $g \in G$ one has the following alternative. Either some positive power of g is conjugate to its inverse, or G admits a homogeneous quasi-morphism which is positive on g. This follows with minor extra efforts from a work by Epstein and Fujiwara (see [EF], proof of Lemma 3.5). Another natural generalization of 1.5.D is as follows. One asks whether there exists a quasi- morphism which attains prescribed values on a given finite subset of the group G. A solution of this problem for discrete subgroups of $PSL(2, \mathbb{R})$ as well as an application to stable mixing of linear maps of the 2-torus will be presented in a forthcoming paper [PR].

APPENDIX: The continuous case

This appendix to §3 is devoted to discussion of the continuous case, that is we take G to be $PSL(2,\mathbb{R})$ acting on $PSL(2,\mathbb{R})/\Gamma$, where Γ is a lattice. We will take the subgroup

$$h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

which gives the horocycle flow on X. Our problem is (cf. 1.8.C above):
Is the horocycle flow on $PSL(2,\mathbb{R})/\Gamma$ stably mixing?

3.3. Quasi-mixing.

First of all, let us relax the mixing property as follows.

Definition 3.3.A. A sequential system f_* acting on a measure space (X,μ) by measure-preserving automorphisms is called *quasi-mixing* if there exists a sequence of positive integers $i_k \to +\infty$ such that for any L^2-functions F and H on X

$$\int_X F(f^{(i_k)}x)H(x)d\mu \longrightarrow \int_X F(x)d\mu \int_X H(x)d\mu$$

when $k \to \infty$. That is, the subsequence $\{f^{(i_k)}\}$ is mixing.

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a lattice. Consider the left action of $PSL(2,\mathbb{R})$ on the space $PSL(2,\mathbb{R})/\Gamma$ endowed with the Haar measure. Write $h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for the horocycle flow, and let $\phi_* = \{\phi_i\}$ be an arbitrary sequence of kicks from $PSL(2,\mathbb{R})$. Denote by $QM(\phi_*)$ the set of those periods $\tau \in (0;+\infty)$ for which the kicked system $\{\phi_i h^\tau\}$ is quasi-mixing.

From general considerations, it follows that a quasi-mixing sequence is *unbounded* (that is has non-compact closure in $PSL(2,\mathbb{R})$). It follows from the Howe-Moore theorem 1.5.A that the converse is also true.

Question 3.3.B. Is it true that for every sequence ϕ_* the set $QM(\phi_*)$ has "large measure"?

Write $\phi_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. We give a partial affirmative answer to Question 3.3.B in terms of the sequence $\{c_i\}$.

Theorem 3.3.C. *If $c_i = 0$ for all i then the set $(0;+\infty)\backslash QM(\phi_*)$ contains at most 1 point.*

Theorem 3.3.D. *Assume that $c_i \neq 0$ for all i and the sequence $\frac{1}{n}\sum_{i=1}^{n} \log|c_i|$ is bounded from below. Then the set $(0;+\infty)\backslash QM(\phi_*)$ has finite measure.*

We prove these theorems in 3.5, 3.6 below.

Interestingly enough, Theorems 3.3.C and 3.3.D handle two opposite cases: when all c_i vanish, and when all c_i are bounded away from 0. At present it is unclear how to attack the intermediate situation.

Remark 3.3.E. Let $M(\phi_*)$ be the set of those periods τ for which the kicked system is mixing (equivalently, goes to infinity in $PSL(2,\mathbb{R})$). In general, one cannot hope that this set has large measure. Here is an example which in fact reflects geometry of real numbers and has nothing to do with the Möbius group. We will produce a sequence of kicks of the form h^{β_i}, where the sequence $\{\beta_i\}$ is chosen as follows. It is not hard to exhibit a sequence of intervals

$$I_k = [r_k; r_k + \frac{1}{k}] \subset [0; +\infty),\ k \in \mathbb{N}$$

which cover every non-negative real number infinitely many times [11].

Put now $\beta_k = (k-1)r_{k-1} - kr_k$, where $r_0 = 0$. One calculates that the evolution of the kicked system is given by $f^{(k)}(\tau) = h^{k(\tau - r_k)}$. Pick up any positive real τ. Note that $\tau \in I_k$ if and only if $k(\tau - r_k) \in [0; 1]$, and thus the last inclusion holds true for infinite number of k due to our choice of intervals I_k. We conclude that for every value of τ the kicked system is neither mixing, nor its evolution goes to infinity. We still have no answer to the following question.

Question 3.3.F. Assume that the sequence of kicks $\phi_* = \{\phi_i\}$ has compact closure in $PSL(2,\mathbb{R})$. Is it true that the set $M(\phi_*)$ has large measure?

3.4. A link to discrete Schrödinger equation.

Question 3.3.B turns out to be nontrivial even in the case when the kicks ϕ_i have a very simple form

$$\phi_i = \begin{pmatrix} 1 & 0 \\ c_i & 1 \end{pmatrix},$$

that is ϕ_i's are time-c_i-maps of the conjugate horocycle flow

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h^{-t} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Fix $\tau > 0$, and write

$$f^{(k)}(\tau) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix},$$

where $f^{(k)}(\tau)$ is the evolution of the kicked system. A straightforward calculation shows that the matrix coefficients satisfy the following recursive relations:

$$\begin{cases} \alpha_k = \alpha_{k-1} + \tau\gamma_{k-1} \\ \gamma_k = \gamma_{k-1} + c_k\alpha_k \end{cases},$$

$$\begin{cases} \beta_k = \beta_{k-1} + \tau\delta_{k-1} \\ \delta_k = \delta_{k-1} + c_k\beta_k \end{cases}.$$

[11] Indeed, one first partitions the divergent series $\sum 1/k$ into infinitely many divergent subseries $a_{m,n}$, $\sum_n a_{m,n} = \infty$ for all m. To do this, first divide the sequence $1/k$ into consecutive blocks so that the sum of elements in each block is at least 1. Then by taking a bijection $j : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ one defines the m-th subsequence as the union of the $j(m,n)$-th blocks, $n = 1, 2, \ldots$.

Now denote by $s_{m,n} = \sum_{t \le n} a_{m,t}$ ($s_{m,0} := 0$) the partial sums of the m-th subsequence, and construct the sequence of intervals $J_{m,n} = [s_{m,n-1}, s_{m,n}]$. Then for each $m \ge 1$, $\cup_{n \ge 1} J_{m,n} = [0, \infty)$ and so we get a sequence of intervals $I_k = [r_k, r_k + 1/k]$ which cover every point infinitely many times.

Both sequences $\{\alpha_k\}$ and $\{\beta_k\}$ satisfy the second order difference equation

$$(3.4.A) \qquad q_{k+1} - (2 + \tau c_k) q_k + q_{k-1} = 0 \quad k \geq 1 \ .$$

Note that this is the discrete Schrödinger equation with a potential, which depends on the parameter τ. Every solution of 3.4.A is uniquely determined by the initial conditions q_0 and q_1. We get the following result.

Proposition 3.4.B. *The sequence $\{f^{(k)}(\tau)\}$, $k \in \mathbb{N}$ is bounded if and only if all the solutions of the Schrödinger equation (3.4.A) are bounded.*

It is instructive to translate Proposition 3.4.B into the language of operator theory. Consider the space V of all real sequences $q = (q_1, q_2, \dots)$. Define linear operators on V,

$$L : (q_1, q_2, \dots, q_i, \dots) \mapsto (c_1 q_1, c_2 q_2, \dots, c_i q_i, \dots)$$

and

$$\Delta_u : (q_1, q_2, \dots, q_i, \dots) \mapsto (u q_1 - q_2, -q_1 + 2q_2 - q_3, \dots, -q_{i-1} + 2q_i - q_{i+1}, \dots) \ .$$

Here $u \in \mathbb{R}$ is a parameter, and the i-th coordinate of $\Delta_u q$ is simply the second difference of the sequence q for all $i \geq 2$. Consider an operator $K_{u,\tau} = \tau L + \Delta_u$. Note that every vector $q \in Ker K_{u,\tau}$ describes a solution of the Schrödinger equation 3.4.A with the boundary condition $q_0 = (2-u) q_1$.

Consider the subspace $V_b \subset V$ consisting of all bounded sequences. With this notation the discussion above leads to the following statement.

Proposition 3.4.C. *The sequence $\{f^{(k)}(\tau)\}$, $k \in \mathbb{N}$ is bounded if and only if $Ker K_{u,\tau} \subset V_b$ for all $u \in \mathbb{R}$.*

The next result is a version of Theorem 3.3.D above.

Proposition 3.4.D. *Suppose that $|c_i| \geq \varepsilon > 0$ for all $i \in \mathbb{N}$. Then there exists $\tau_0 > 0$ such that the sequence $\{f^{(k)}(\tau)\}$ is unbounded for all $\tau > \tau_0$.*

Proof: Fix $u \in \mathbb{R}$. We claim that for τ large enough $Ker K_{u,\tau} \cap V_b = \{0\}$. Assume the claim. Since $\dim Ker K_{u,\tau} = 1$ we get that $Ker K_{u,\tau}$ is not contained in V_b for τ large enough. Thus the desired result follows from 3.4.C.

It remains to prove the claim. Endow the space V_b with the norm $\|q\| = \sup_i |q_i|$. Our assumption on c_i implies that operator L is invertible, $L^{-1}(V_b) = V_b$ and $\|L^{-1}\| \leq \frac{1}{\varepsilon}$. Further, $\Delta_u(V_b) \subset V_b$, and Δ_u is a bounded operator. Denote $\|\Delta_u\| = v$. We have to solve the equation $K_{u,\tau} q = 0$, $q \in V_b$. Note then that $(\tau L + \Delta_u) q = 0$, that is $(\mathbb{1} + \tau^{-1} L^{-1} \Delta_u) q = 0$. Since $\|L^{-1} \Delta_u\| \leq v/\varepsilon$ we see that the operator $\mathbb{1} + \tau^{-1} L^{-1} \Delta_u$ is invertible for $\tau > v/\varepsilon$, and therefore $q = 0$. This proves the claim. □

The proof above illustrates the difficulty which one faces in the case when the coefficients c_i are allowed to approach arbitrarily close to 0. Indeed, the operator $L^{-1}\Delta_u$ becomes unbounded, and one loses control on the kernel of $\mathbb{1} + \tau^{-1} L^{-1} \Delta_u$ even for large values of τ.

Let us present two additional cases when one gets the affirmative answer to Question 3.3.B assuming that $\phi_i = \begin{pmatrix} 1 & 0 \\ c_i & 1 \end{pmatrix}$.

(3.4.E) $c_i \to 0$ when $i \to \infty$;

(3.4.F) all c_i are non-negative.

Indeed assume that $c_i \to 0$. We write $|\psi|$ for the Euclidean norm of a matrix $\psi \in PSL(2, \mathbb{R})$, $|\psi| = \sqrt{tr\psi\psi^*}$. Let us show that the sequence $|f^{(k)}(\tau)|$ is unbounded for every $\tau > 0$. Assume on the contrary that for some $\tau > 0$ holds $|f^{(k)}(\tau)| \leq K$ for all $k \in \mathbb{N}$. Since $c_i \to 0$ there exists $i, j > 0$ such that

$$|\phi_{i+j} h^\tau \cdots \phi_{i+1} h^\tau| \geq 2K^2 \ .$$

But

$$|\phi_{i+j} h^\tau \cdots \phi_{i+1} h^\tau| = |f^{(i+j)}(\tau) \cdot (f^{(i)}(\tau))^{-1}| \leq K^2 \ .$$

This contradiction proves the claim. [12]

Assume now that all c_i are non-negative, and for some $\tau > 0$ the sequence $|f^{(k)}(\tau)|$ is bounded. Write $f^{(k)}(\tau) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}$, and note that our assumption implies that the matrix coefficients are non-negative and bounded above. The recursive relations listed in the beginning of this section show that the sequence $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ are non-decreasing, and thus they converge to some values $\alpha_\infty, \beta_\infty, \gamma_\infty, \delta_\infty$. Since the matrix $\begin{pmatrix} \alpha_\infty & \beta_\infty \\ \gamma_\infty & \delta_\infty \end{pmatrix}$ belongs to $PSL(2, \mathbb{R})$ we have that either $\alpha_\infty \neq 0$ or $\beta_\infty \neq 0$. Assume without loss of generality that $\alpha_\infty \neq 0$. Since $c_k = \frac{\gamma_k - \gamma_{k-1}}{\alpha_k}$ we conclude that $c_k \to 0$ when $k \to \infty$, and we are in the case (3.4.E) considered above. This completes the analysis of (3.4.E) and (3.4.F).

3.5. Proof of Theorem 3.3.C.

Write $\phi_k = \begin{pmatrix} a_k & b_k \\ 0 & a_k^{-1} \end{pmatrix}$. Fix $k > 0$ and for $m \leq k$ denote $\psi_m = \phi_k \cdot \ldots \cdot \phi_m$. Then the evolution of the kicked systems can be written as follows:

$$(3.5.A) \qquad f^{(k)}(\tau) = \phi_k h^\tau \cdot \ldots \cdot \phi_1 h^\tau = \Big(\prod_{j=0}^{k-1} \psi_{k-j} h^\tau \psi_{k-j}^{-1} \Big) \cdot \psi_1 \ .$$

Note that if $\psi = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$, then

$$(3.5.B) \qquad \psi h^\tau \psi^{-1} = \begin{pmatrix} 1 & \tau u^2 \\ 0 & 1 \end{pmatrix} \ .$$

The matrix ψ_1 has the form $\begin{pmatrix} a_1 \cdot \ldots \cdot a_k & w_k \\ 0 & a_1 \cdot \ldots \cdot a_k \end{pmatrix}$, where w_k is some real number. Substituting this to (3.5.A) and using (3.5.B) we get the following expression for the evolution

[12] This argument was suggested to us by D. Kazhdan.

of the kicked system:

$$f^{(k)}(\tau) = \begin{pmatrix} a_1 \cdot \ldots \cdot a_k & w_k + \tau z_k \\ 0 & a_1^{-1} \cdot \ldots \cdot a_k^{-1} \end{pmatrix} , \tag{3.5.C}$$

where

$$z_k = \frac{\sum_{i=1}^{k} (a_k \cdot \ldots \cdot a_i)^2}{a_1 \cdot \ldots \cdot a_k}.$$

Now assume that for some τ_0, the sequence $\{f^{(k)}(\tau)\}, k \in \mathbb{N}$ is bounded. Then there exist constants $\alpha > \beta > 0$ such that $\beta \le |a_1 \cdots a_k| \le \alpha$ for all k (look at the diagonal terms of 3.5.C). Therefore, for every k and i, we have $|a_k \cdot \ldots \cdot a_i| \ge \beta\alpha^{-1}$ and thus

$$|z_k| \ge \beta^2 \alpha^{-3} k \ .$$

Note that

$$f^{(k)}(\tau) - f^{(k)}(\tau_0) = \begin{pmatrix} 0 & r_k(\tau) \\ 0 & 0 \end{pmatrix} ,$$

where $r_k(\tau) = z_k(\tau - \tau_0)$. Since $|r_k(\tau)| \ge \beta^2\alpha^{-3}k|\tau - \tau_0|$, we conclude that the sequence $f^{(k)}(\tau)$ is unbounded for every $\tau \ne \tau_0$. This completes the proof. □

3.6. Proof of Theorem 3.3.D.

We start with the following

Lemma 3.6.A. *Let p_k be a sequence of real polynomials of degree n with leading coefficients α_k. Suppose that $|\alpha_k| \ge \lambda^k$ for some $\lambda > 0$. Then the set*

$$Y = \{\tau \in \mathbb{R} | \text{ the sequence } \{p_k(\tau)\} \text{ is bounded}\}$$

has finite Lebesgue measure.

Proof of 3.3.D: Write $\phi_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$, and put $p_k(\tau) = \text{trace } (f^{(k)}(\tau))$, where $f^{(k)}(\tau) = \phi_k h^\tau \cdot \ldots \cdot \phi_1 h^\tau$. It is easy to see that $p_k(\tau)$ is a degree k polynomial with the leading coefficient $c_1 \cdot \ldots \cdot c_k$. Our assumption on the sequence $\{c_i\}$ implies that $|c_1 \ldots c_k| \ge \lambda^k$ for some $\lambda > 0$. Applying Lemma 3.6.A, we get the statement of the Theorem. □

Proof of 3.6.A: Take $T > 0$, and write $Y_T = Y \cap [-T; T]$. The sequence $\frac{1}{k}p_k(\tau)$ converges to 0 when $k \to \infty$ for every $\tau \in Y_T$. Applying the Egorov theorem [FW] we get that there exists a subset $Z_T \subset Y_T$ such that

$$\text{measure} Z_T \ge \frac{1}{2} \text{measure} Y_T$$

and the sequence $\frac{1}{k}p_k(\tau)$ converges *uniformly* on Z_T. In particular, there exists $k_0 > 0$ such that $|p_k(\tau)| \le k$ for all $\tau \in Z_T$ and $k \ge k_0$. Setting $\widetilde{p}_k(\tau) = \alpha_k^{-1} p_k(\tau)$, this implies that

$$|\widetilde{p}_k(\tau)| \le k|\alpha_k|^{-1} \quad \text{for all} \quad \tau \in Z_T \ . \tag{3.6.B}$$

Note that $\widetilde{p}_k(\tau)$ is a polynomial of degree k with the leading coefficient 1. A theorem due to Polya (see [T], 2.9.13) states that for any measurable subset $Z \subset \mathbb{R}$

$$(3.6.C)\ . \qquad \max_{\tau \in Z} |\widetilde{p}_k(\tau)| \geq 2 \cdot \left(\frac{\text{measure } Z}{4}\right)^k$$

Substituting $Z = Z_T$ and combining with (3.6.B) we get

$$k|\alpha_k|^{-1} \geq 2 \cdot \left(\frac{\text{measure } Z_T}{4}\right)^k \geq 2 \cdot \left(\frac{\text{measure } Y_T}{8}\right)^k .$$

Since $|\alpha_k| \geq \lambda^k$ we obtain that

$$k \geq 2 \cdot \left(\frac{\text{measure } Y_T}{8} \cdot \lambda\right)^k .$$

This inequality holds for every $k \geq k_0$, so

$$\text{measure } Y_T \leq \frac{8}{\lambda} .$$

Since this is true for every $T > 0$ we conclude that

$$\text{measure } Y \quad \leq \frac{8}{\lambda} .$$

This completes the proof. $\square$

4. Stable super-recurrence in Hamiltonian dynamics

In the present section we prove slightly more general versions of Theorems 1.6.A,B,C. Recall that these theorems provide a sufficient condition for kick stable energy conservation law and super-recurrence in terms of Hofer's geometry of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X,\Omega)$. In practice, it is easier to perform measurements not on $\mathrm{Ham}(X,\Omega)$ itself but on its universal cover. It turns out that these simpler measurements are powerful enough to detect stable super-recurrence.

At the very end of the section we prove Proposition 1.7.F stated in the Introduction.

4.1. Dynamical preliminaries

In this section we describe a link between quasi-integrals (see 1.1.C) and super-recurrence (see 1.1.D) in the context of general sequential systems. Let X be a compact topological space endowed with a Borel probability measure μ. Let $f_* = \{f_i\}$ be a sequential system which acts on X by μ-preserving homeomorphisms. Assume that f_* is not strictly ergodic (see 1.1.B). Then there exists a continuous function F on X with zero mean such that $F \not\equiv 0$ and

$$\limsup_{N\to\infty} \max \frac{1}{N}\sum_{i=0}^{N-1} F\circ f^{(i)} \geq \alpha \max F \tag{4.1.A}$$

for some $\alpha \in (0;1]$. In this case we say that F is an *α-quasi-integral* of f_*.

For an open subset $A \subset X$ and $N \in \mathbb{N}$ define the counting function $\nu_{N,A} : X \to \mathbb{R}$ as follows. For $x \in X$ put $\nu_{N,A}(x)$ to be the the cardinality of the set

$$\{i \in [0;N-1] \mid f^{(i)}(x) \in A\}.$$

Define the quantity

$$R(f_*,A) = \limsup_{N\to+\infty} \max_{x\in X} \frac{1}{N}\nu_{N,A}(x).$$

For every $\epsilon > 0$ there exist arbitrarily long finite pieces $\{x_0 = x, ..., x_{N-1}\}$ of trajectories of f_* which visit A with the frequency at least $R(f_*,A)-\epsilon$. Clearly, A is super-recurrent if $R(f_*,A)$ is strictly bigger than $\mu(A)$.

It turns out that in some situations one can extract fairly explicit information on super-recurrent sets from quasi-integrals. Let F be an α-quasi-integral of f_*. Put $\gamma = \left|\frac{\min F}{\max F}\right|$, and denote by $A_c, c \in (0;1)$ the set

$$\{x \in X \mid F(x) \geq c \max F\}\ .$$

Theorem 4.1.B. *Suppose that $\gamma < \frac{\alpha^2}{4-4\alpha}$. Then for every $c \in (0;1)$ which satisfies*

$$\left|c - \frac{\alpha}{2}\right| < \frac{1}{2}\sqrt{a^2 + 4\alpha\gamma - 4\gamma}$$

the set A_c is super-recurrent for the system f_*. Moreover, $R(f_*,A_c) - \mu(A_c) \geq \frac{\alpha-c}{1-c} - \frac{\gamma}{c+\gamma}$.

The proof is based on the following lemmas.

Lemma 4.1.C. *For every $c \in (0;\alpha)$*

$$R(f_*, A_c) \geq \frac{\alpha - c}{1 - c} \ .$$

Proof. Choose arbitrary $\varepsilon > 0$. Denote

$$I_N = \frac{1}{N} \sum_{i=0}^{N-1} F \circ f^{(i)} \ .$$

There exists an arbitrarily large positive integer N such that for some $x_0 \in X$

$$I_N(x_0) \geq (\alpha - \varepsilon) \max F \ .$$

Write ν_N for the counting function ν_{N,A_c} defined in the beginning of this section. Clearly,

$$N I_N(x_0) \leq c \max F (N - \nu_N(x_0)) + \max F \cdot \nu_N(x_0) \ .$$

Combining this with the previous inequality we get that

$$\max_x \frac{\nu_N(x)}{N} \geq \frac{\alpha - c - \varepsilon}{1 - c} \ .$$

Since this holds true for all $\varepsilon > 0$ and for an infinite sequence of positive values of N we conclude that

$$R(f_*, A_c) = \limsup_{N \to +\infty} \frac{\nu_N(x)}{N} \geq \frac{\alpha - c}{1 - c} \ .$$

□

Lemma 4.1.D. *For every $c \in (0;\alpha)$*

$$\mu(A_c) \leq \frac{\gamma}{c + \gamma} \ .$$

Proof. Note that

$$0 = \int_X F d\mu = \int_{A_c} F d\mu + \int_{X \setminus A_c} F d\mu \geq c \max F \cdot \mu(A_c) + \min F \cdot (1 - \mu(A_c)) \ .$$

Thus

$$\mu(A_c) \leq \frac{-\min F}{c \max F - \min F} = \frac{\gamma}{c + \gamma} \ .$$

□

Proof of 4.1.B. The assumptions of the theorem guarantee that $\frac{\alpha - c}{1-c} > \frac{\gamma}{c+\gamma}$. Applying 4.1.C and 4.1.D we get that

$$R(f_*, A_c) - \mu(A_c) \geq \frac{\alpha - c}{1 - c} - \frac{\gamma}{c + \gamma} > 0 \ .$$

This completes the proof. □

4.2 Geometric preliminaries

Let (X, Ω) be a closed symplectic manifold. For a smooth path $q^t, t \in [a;b]$ of Hamiltonian diffeomorphisms of (X, Ω) set

$$\text{length } (q^t) = \int_a^b \max_{x \in X} Q(x,t)dt \ ,$$

where Q is the normalized Hamiltonian generating the path. Here we use the following normalization:

$$\int_X Q(x,t)d\mu = 0$$

for all $t \in [a,b]$. Let (h^t) be a one parameter subgroup of $\text{Ham}(X,\Omega)$ generated by the time-independent Hamiltonian H. Define a function $\ell_H : [0; +\infty) \to [0; +\infty)$ by

$$\ell_H(s) = \inf \text{length } (q^t) \ ,$$

where the infimum is taken over all paths (q^t), $t \in [0;s]$ with the following properties:

- $q_0 = \mathbb{1}, q^s = h^s$;
- the paths q^t and h^t are homotopic through smooth paths with fixed end points.

Clearly,

$$(4.2.A) \qquad s \max H \geq \ell_H(s) \geq \rho(h^s) \ ,$$

where ρ is the positive path of Hofer's norm defined in 1.6. It is easy to see that $\ell_H(s+t) \leq \ell_H(s) + \ell_H(t)$ for all $s,t > 0$, so the limit

$$\ell_\infty(H) = \lim_{S \to \infty} \frac{\ell_H(s)}{s \max H}$$

exists. This quantity always belongs to the unit segment $[0;1]$.

In a number of interesting situations one can find non-trivial lower bounds for $\ell_\infty(H)$ using tools of modern symplectic topology. Here we present such a bound (see 4.2.D below) which was obtained in [P1],[P4]. Recall that a submanifold $L \subset (X, \Omega)$ is called Lagrangian if $\dim L = \frac{1}{2} \dim X$, and the symplectic form Ω vanishes on TL.

Definition 4.2.B. Let $L \subset X$ be a closed Lagrangian submanifold. We say that L has *the Lagrangian intersection property* if $L \cap \phi(L) \neq \emptyset$ for every Hamiltonian diffeomorphism $\phi \in \text{Ham}(X, \Omega)$.

For example, the *equator* of the 2-sphere (that is a simple closed curve which divides the sphere into two discs of equal areas) clearly has the Lagrangian intersection property.

Consider the cylinder T^*S^1 endowed with coordinates $r \in \mathbb{R}$ and $t \in S^1$. The standard symplectic form on T^*S^1 is written as $dr \wedge dt$. Denote by Z the zero section $\{r = 0\}$. For a symplectic manifold (X, Ω) consider the topological stabilization $(X \times T^*S^1, \Omega + dr \wedge dt)$. If L is a closed Lagrangian submanifold of X then $L \times Z$ is a closed Lagrangian submanifold of $X \times T^*S^1$.

Definition 4.2.C. Let $L \subset X$ be a closed Lagrangian submanifold. We say that L has *the stable Lagrangian intersection property* if $L \times Z$ has the Lagrangian intersection property in $X \times T^*S^1$.

Remark: It is easily seen that stable Lagrangian intersection property implies the Lagrangian intersection property.

In many situations, one can detect the stable Lagrangian intersection property with the help of the Floer homology. Let us give two examples.

- A closed Lagrangian submanifold $L \subset X$ with $\pi_2(X, L) = 0$ has stable Lagrangian intersection property;
- The equator of the 2-sphere, which trivially has the Lagrangian intersection property, can be shown to in fact have the *stable* Lagrangian intersection property.

We refer to [P1],[P4] for further details and references.

Theorem 4.2.D. [P1],[P4] *Let (h^t) be a one parameter subgroup of $\mathrm{Ham}(X, \Omega)$ generated by a normalized Hamiltonian function H. Assume that there exists a closed Lagrangian submanifold $L \subset X$ with stable Lagrangian intersection property such that $H(x) \geq C > 0$ for all $x \in L$. Then $\ell_H(s) \geq Cs$ for all $s > 0$, and $\ell_\infty(H) \geq \frac{C}{\max H}$.*

4.3. Detecting stable super-recurrence

Let (X, Ω) be a closed symplectic manifold. Let (h^t) be a one parameter subgroup of $\mathrm{Ham}(X, \Omega)$ generated by a time-independent normalized Hamiltonian H. Take an arbitrary bounded sequence $\phi_* = \{\phi_i\}$ of Hamiltonian diffeomorphisms of (X, Ω), and consider the kicked system $f_*^\tau = \{\phi_i h^\tau\}$. Put

$$A_c = \{x \in X \mid H(x) > c \max H\} \ ,$$

where $c \in (0; 1)$.

Theorem 4.3.A. *Suppose that $\ell_H(s) = s \max H$ for all $s > 0$. Then for every $c, \alpha \in (0; 1)$ and $\varepsilon > 0$ there exists a subset $P \subset (0; \infty)$ with the following properties:*

- *the set $(0; +\infty) \backslash P$ has finite Lebesgue measure;*
- *for every $\tau \in P$ the Hamiltonian H is an α-quasi-integral of the kicked system f_*^τ;*
- *for every $\tau \in P$ the set A_c is super-recurrent for the kicked system f_*^τ with $R(f_*^\tau, A_c) > 1 - \varepsilon$.*

Proof of Theorem 1.2.D: Since the maximum set of H contains an equator, and the equator has the stable Lagrangian intersection property (see 4.2 above) we conclude from 4.2.D and 4.2.A that $\ell_H(s) = s \max H$ for all s. The Theorem follows now from 4.3.A. □

The condition $\ell_H(s) = s \max H$ is very restrictive. However our technique enables us to detect a weaker version of kick stable super-recurrence in a more general situation. Recall that the *density* of a subset $P \subset (0; +\infty)$ is

$$\liminf_{T \to +\infty} \frac{1}{T} \text{ measure } (P \cap (0; T]) \ .$$

Suppose that $\ell_\infty(H) > 0$ and choose any $\alpha \in (0; \ell_\infty(H))$. Put

$$\theta = \frac{\ell_\infty(H) - \alpha}{1 - \alpha}. \tag{4.3.B}$$

Theorem 4.3.C. *There exists a subset $P \subset (0; +\infty)$ of density at least θ such that for every $\tau \in P$ the Hamiltonian H is an α-quasi-integral of the kicked system f_*^τ.*

Theorems 4.3.A and 4.3.C are proved in 4.5 below.

Proof of Theorem 1.6.A: The Theorem follows from 4.3.C. Indeed, $\theta \to \ell_\infty(H)$ when $\alpha \to 0$. □

Let us describe an application of Theorem 4.3.C to stable super-recurrence. Set $\gamma = \left|\frac{\min H}{\max H}\right|$. We assume that the following inequality holds:

$$\gamma(4 - 4\ell_\infty(H)) < \ell_\infty(H)^2 \ . \tag{4.3.D}$$

Denote by $P_{c,\delta}$ (where $c, \delta \in (0; 1)$) the set of those values of the period $\tau \in (0; +\infty)$ such that the set A_c is super-recurrent for the kicked system f_*^τ with $R(f_*^\tau, A_c) - \mu(A_c) > \delta$. Choose arbitrary $\alpha \in (0; \ell_\infty(H))$ such that

$$\gamma < \frac{\alpha^2}{4 - 4\alpha}.$$

Pick any $c \in (0; 1)$ which satisfies

$$\left| c - \frac{\alpha}{2} \right| < \frac{1}{2}\sqrt{\alpha^2 + 4\alpha\gamma - 4\gamma} \ . \tag{4.3.E}$$

Set

$$\delta = \frac{\alpha - c}{1 - c} - \frac{\gamma}{c + \gamma} \tag{4.3.F}$$

One concludes from (4.3.E) that δ is positive. The next result follows immediately from 4.3.C and 4.1.B. Here we assume (4.3.D), and define θ by (4.3.B).

Theorem 4.3.G. *The density of $P_{c,\delta}$ is greater or equal to θ.*

Proof of Theorem 1.6.B. It follows from 4.2.A that $\ell_\infty(H) = 1$. Thus assumption 4.3.D holds. Put $c = \epsilon$ and choose α so close to 1 that 4.3.E holds. Note that $\theta = 1$. Define δ by 4.3.F. It follows from 4.3.G that A_c is super-recurrent when τ belongs to the set $P_{c,\delta}$ of density 1. □

Proof of Theorem 1.6.C. It follows from (4.2.A) that $\ell_\infty(H) \geq 0.9$. Further, it is given that $\gamma = 1$. Thus assumption 4.3.D holds. Choose $\alpha = 0.83$, and $c = 0.4$. Note that the inequality (4.3.E) is satisfied and $\theta = \frac{0.9-0.83}{1-0.83} > 0.4$. Theorem follows now from 4.3.G. □

4.4. A geometric inequality

The main ingredient of our approach to Theorems 4.3.A and 4.3.C is the following upper bound on the function $\ell_H(s)$. Let (h^t) be a one parameter subgroup of $\mathrm{Ham}(X, \Omega)$ generated by a time-independent Hamiltonian H with zero mean. Take an arbitrary sequence $\{\phi_i\}$ of Hamiltonian diffeomorphisms. Denote by $f^{(i)}(\tau)$ the evolution of the kicked system, that is

$$f^{(i)}(\tau) = \phi_i h^\tau \phi_{i-1} h^\tau \cdot \ldots \cdot \phi_1 h^\tau \ ,$$

where $i \geq 1$, $\tau \in (0; \infty)$. Put $f^{(0)}(\tau) \equiv \mathbb{1}$. We write $\bar{\rho}$ for Hofer's norm defined 1.2.C.

Theorem 4.4.A. *For every $N \in \mathbb{N}$ and $T > 0$ holds*

$$\ell_H(NT) \leq \int_0^T \max_X \sum_{i=0}^{N-1} H \circ f^{(i)}(t) dt + 2 \sum_{i=1}^{N-1} \bar{\rho}(\phi_i) \ .$$

Proof: The proof is divided into several steps.

1) Decompose $h^{NT} = A_{N,T} B_{N,T}$, where

$$A_{N,T} = h^T \cdot \prod_{i=1}^{N-1} \psi_i h^T \psi_i^{-1} \ ,$$

with

$$\psi_i = \psi_{i,N} = \phi_{N-1} \cdot \ldots \cdot \phi_{N-i} \ , \quad i = 1, \ldots, N-1$$

and

$$B_{N,T} = \phi_{N-1} \cdot \ldots \cdot \phi_1 \prod_{j=1}^{N-1} h^{-jT} \phi_j^{-1} h^{jT} \ .$$

Take $\varepsilon > 0$, and choose paths $\phi_i^{(s)}$, $s \in [0; T]$ of Hamiltonian diffeomorphisms which join $\mathbb{1}$ with ϕ_i so that the lengths of the paths $\phi_i^{(s)}$ and $(\phi_i^{(s)})^{-1}$ do not exceed $\bar{\rho}(\phi_i) + \varepsilon$ for all i. Consider the following paths of Hamiltonian diffeomorphisms defined for $s \in [0; T]$:

$$c_s = h^{Ns}$$

$$a_s = h^s \prod_{i=1}^{N-1} \psi_i h^s \psi_i^{-1}$$

and

$$b_s = \phi^{(s)}_{N-1} \cdots \phi^{(s)}_1 \prod_{j=1}^{N-1} h^{-jT} (\phi^{(s)}_j)^{-1} h^{jT} \ .$$

The paths $\{c_s\}$ and $\{a_s b_s\}$ join $1\!\!1$ with h^{NT}.

2) We claim that the paths $\{c_s\}$ and $\{a_s b_s\}$ are homotopic with fixed endpoints. Indeed take the parameter of homotopy $u \in [0;1]$ and write

$$a_{s,u} = h^s \prod_{i=1}^{N-1} \psi^{(u)}_i h^s (\psi^{(u)}_i)^{-1} \ ,$$

where $\psi^{(u)}_i = \phi^{(Tu)}_{N-1} \cdots \phi^{(Tu)}_{N-i}$. Set

$$b_{s,u} = \phi^{(su)}_{N-1} \cdots \phi^{(su)}_1 \prod_{j=1}^{N-1} h^{-jT} (\phi^{(su)}_j)^{-1} h^{jT} \ .$$

The required homotopy is given by $d_{s,u} = a_{s,u} b_{s,u}$ where $s \in [0;T]$ and $u \in [0;1]$.

3) It follows from the definition of the function ℓ_H that $\ell_H(NT) \leq \text{length}(a_s b_s)_{s \in [0;T]}$. But length $(a_s b_s) \leq \text{length}(a_s) + \text{length}(b_s)$, and

$$\text{length}(b_s) \leq 2 \sum_{j=1}^{N-1} \bar\rho(\phi_j) + 2(N-1)\varepsilon$$

(one uses here that Hofer's norm $\bar\rho$ is bi-invariant). We conclude that

$$\ell_H(NT) \leq \text{length}(a_s)_{s\in[0;T]} + \\ +2 \sum_{j=1}^{N-1} \bar\rho(\phi_j) + 2(N-1)\varepsilon. \tag{4.4.B}$$

4) Denote by $\widetilde{F}(x,s) = \widetilde{F}_s(x)$ the normalized Hamiltonian function generating (a_s). In order to calculate $\widetilde{F}$, we use the following product formula: Let $(p_s), (q_s), s \in [0;T]$ be two Hamiltonian flows generated by normalized Hamiltonians $P(x,s)$ and $Q(x,s)$. Then the product $(p_s q_s)$ is a Hamiltonian flow generated by normalized Hamiltonian $P(x,s) + Q(p_s^{-1} x, s)$. In particular, for a given Hamiltonian diffeomorphism ψ the path $(\psi h^s \psi^{-1})$ is generated by Hamiltonian $H \circ \psi^{-1}$.

Applying the product formula we get that

$$\widetilde{F}_s = H + \sum_{i=1}^{N-1} H \circ \psi_i^{-1} \circ \Big(\prod_{j=0}^{i-1} \psi_j h^s \psi_j^{-1} \Big)^{-1} \ ,$$

where $\psi_0 = 1\!\!1$. Using that $\psi_{k-1}^{-1}\psi_k = \phi_{N-k}$ we can rewrite this as

$$\widetilde{F}_s = H + \sum_{i=1}^{N-1} H \circ \Big(\prod_{j=1}^{i} h^s \phi_{N-j} \Big)^{-1} \ .$$

Introduce a new function

$$F_s = \widetilde{F}_s \circ \Big(\prod_{j=1}^{N-1} h^s \phi_{N-j} \Big) \circ h^s \ .$$

Then

$$F_s = H \circ h^s \circ \Big(\prod_{j=1}^{N-1} \phi_{N-j} h^s \Big) +$$
$$+ \sum_{i=1}^{N-2} H \circ \Big(\prod_{j=i+1}^{N-1} h^s \phi_{N-j} \Big) \circ h^s + H \circ h^s \ .$$

The energy conservation law implies that $H \circ h^s = H$. Thus

$$F_s = H \circ \prod_{j=1}^{N-1} \phi_{N-j} h^s +$$
$$+ \sum_{i=1}^{N-2} H \circ \phi_{N-i-1} h_s \phi_{N-i-2} h_s \cdots \phi_1 h_s + H \ .$$

The last expression can be rewritten in terms of the kicked system:

$$F_s = \sum_{k=0}^{N-1} H \circ f^{(k)}(s) \ .$$

Since $\max\limits_X F_s = \max\limits_X \widetilde{F}_s$ for all s, we get that

$$\mathrm{length}(a_s)_{s\in[0;T]} = \int_0^T \max_X \widetilde{F}_s ds =$$
$$= \int_0^T \max_X \sum_{k=0}^{N-1} H \circ f^{(k)}(s) ds \ .$$

Substituting this expression into (4.4.B) we get the desired inequality 4.4.A. This completes the proof.

4.5 Proof of main theorems

Proof of Theorem 4.3.C. Set $H_N(t) = \frac{1}{N} \max\limits_X \sum\limits_{i=0}^{N-1} H \circ f^{(i)}(t)$. Let P be the set of those $t \in (0;\infty)$ for which the inequality $H_N(t) \geq \alpha \max H$ holds for an infinite sequence of positive integers N. We have to show that the density of P is at least $\theta = \frac{\ell_\infty(H)-\alpha}{1-\alpha}$.

Since the sequence $\{\phi_i\}$ is bounded, there exists $u > 0$ such that $2\bar{\rho}(\phi_i) \leq u$ for all i. Fix $\kappa > 0$, and abbreviate $\ell = \ell_\infty(H)$. There exist $T_0 > 0$, $N_0 > 0$ such that for all $N > N_0$, $T > T_0$

$$\ell_H(NT) \geq NT(\ell - \kappa) \max H \ .$$

The geometric inequality 4.4.A yields

$$\ell_H(NT) \leq N \int_0^T H_N(t)dt + (N-1)u \ .$$

Therefore

$$NT(\ell - \kappa) \max H \leq N \int_0^T H_N(t)dt + Nu \ .$$

Increasing T_0 we can assume that $u \leq \kappa T \max H$, so we get

$$\int_0^T H_N(t)dt \geq (\ell - 2\kappa) T \max H \ . \tag{4.5.A}$$

Consider the set

$$Q_{N,T} = \{t \in [0;T] \mid H_N(t) < \alpha \cdot \max H\} \ .$$

Clearly,

$$\begin{aligned} \int_0^T H_N(t)dt = \int_{Q_{N,T}} H_N(t)dt + \int_{[0;T]\setminus Q_{N,T}} H_N(t)dt \leq \\ \leq \alpha \max H \cdot \text{measure}(Q_{N,T}) + \max H \cdot (T - \text{measure}(Q_{N,T})) \ . \end{aligned}$$

Combining this with 4.5.A we get that

$$\frac{\text{measure}(Q_{N,T})}{T} \leq \frac{1 - \ell + 2\kappa}{1 - \alpha} \tag{4.5.B}$$

Clearly, $[0;T]\setminus P = \bigcup_{k=1}^{\infty} \bigcap_{N=k}^{\infty} Q_{N,T}$, so in view of (4.5.B)

$$\frac{1}{T}\text{measure}(P \cap [0;T]) \geq \frac{\ell - \alpha - 2\kappa}{1 - \alpha}$$

for $T > T_0$. Thus density$(P) \geq \theta = \frac{\ell-\alpha-2\kappa}{1-\alpha}$. Since κ can be chosen arbitrary small, this completes the proof of the theorem. □

Proof of Theorem 4.3.A The proof is analogous to the one of 4.3.C. Take $\alpha \in (0;1)$ and define $H_N(t)$, u and $Q_{N,T}$ as above. Since $\ell_H(NT) = NT \max H$ due to our assumption, and

$$\ell_H(NT) \leq N \int_0^T H_N(t)dt + Nu$$

in view of 4.4.A, we get that

$$\int_0^T H_N(t)dt \geq T \max H - u \ .$$

Therefore

$$\text{measure}(Q_{N,T}) \le \frac{u}{(1-\alpha)\max H} .$$

The set $Q = \bigcup_T \bigcup_{k=1}^{\infty} \bigcap_{N=k}^{\infty} Q_{N,T}$ has finite measure. Consider its complement $P = (0;+\infty)\backslash Q$. For every $\tau \in P$ function H is an α-quasi-integral of the kicked system f_*^τ.

Assume without loss of generality that α is sufficiently close to 1 so that the following hold:

- $\left|c - \frac{\alpha}{2}\right| < \frac{1}{2}\sqrt{\alpha^2 + 4\alpha\gamma - 4\gamma}$;
- $\frac{\alpha-c}{1-c} \ge 1-\varepsilon$, $\frac{\alpha-c}{1-c} > \frac{\gamma}{c+\gamma}$.

Here γ stands for $\left|\frac{\min H}{\max H}\right|$, and ε is given in the formulation of 4.3.A. It follows from 4.1.B that the set $A_c = \{x \in X \mid H(x) \ge c \max H\}$ is super-recurrent for f_*^τ. Moreover, Lemma 4.1.C guarantees that $R(f_*^\tau, A_c) \ge 1-\varepsilon$. This completes proof. □

4.6 Creating random behaviour

Here we prove Proposition 1.7.F of the introduction. Let (X,Ω) be a closed symplectic manifold such that $\text{Ham}(X,\Omega)$ is C^∞-closed in $\text{Diff}(X)$. Note that under this assumption there exists a bounded strictly ergodic sequence $\psi_* = \{\psi_i\}$ of Hamiltonian diffeomorphisms. This is an immediate consequence of [P3], Th. 1.2.A.

Let θ be a time reversing symmetry for (h^t), that is $\theta h^t \theta^{-1} = h^{-t}$ for all $t \in \mathbb{R}$. Define a sequence of kicks $\{\phi_i\}$ as follows. Set $\phi_i = \theta^{-1}$ when i is odd, and $\phi_i = \psi_k\theta$ when i is even and equals $2k$. We claim that for every $\tau > 0$ the kicked system $f_*^\tau = \{\phi_i h^\tau\}$ is strictly ergodic. Indeed, note that $f^{(i)}(\tau) = \phi_i h^\tau \cdot \ldots \phi_1 h^\tau$ equals $\psi^{(k)}$ for $i = 2k$, and $\theta^{-1}h^\tau\psi^{(k)}$ for $i = 2k+1$. For every continuous function F on X set $I_N = \sum_{i=0}^{N-1} F \circ f^{(i)}(\tau)$. We see that for $N = 2k$, $I_N = \sum_{i=0}^{k-1} F \circ \psi^{(i)} + \sum_{i=0}^{k-1} (F \circ \theta^{-1}h^\tau) \circ \psi^{(i)}$. Thus the uniform limit $\lim_{N\to\infty} \frac{1}{N} I_N$ exists and equals $\frac{1}{2}\left(\int_X F d\mu + \int_X F \circ \theta^{-1}h^\tau d\mu\right) = \int_X F d\mu$. Therefore f_*^τ is strictly ergodic for every τ. This completes the proof. □

Example 4.6.A We conclude this section with an example in the spirit of 1.7.G and 1.7.H. It shows that the phenomenon presented in Theorem 4.3.A is a purely Hamiltonian one and may disappear when one allows kicks which are symplectic but not necessarily Hamiltonian diffeomorphisms of (X,Ω). Consider the 2-torus $\mathbb{T}^2 = \mathbb{R}^2(x,y)/\mathbb{Z}^2$ endowed with the symplectic form $\Omega = dx \wedge dy$. Consider a one parameter subgroup (h^t) of Hamiltonian diffeomorphisms given by

$$h^t(x,y) = (x, y - t\sin 2\pi x) .$$

It is generated by a normalized Hamiltonian function $H(x,y) = \frac{1}{2\pi}\cos 2\pi x$, which attains its maximal value on a non-contractible curve $L = \{x = 0\}$. Since L has stable Lagrangian intersection property (see 4.2.C above) it follows from Theorem 4.2.D that $\ell_H(s) \equiv s \max H$. Thus Theorem 4.3.A implies stable super-recurrence. On the other hand the shift $\theta : (x,y) \mapsto (x + \frac{1}{2}, y)$ is a time reversing symmetry of (h^t). Note that θ

is a symplectic, but not a Hamiltonian diffeomorphism. Our proof above shows that there exists a sequence of symplectomorphisms $\{\phi_i\}$ such that the kicked system $\{\phi_i h^\tau\}$ is strictly ergodic for each value of τ.

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